

Review on High energy String Scattering Amplitudes and Symmetries of String Theory

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Abstract

We review high energy symmetries of string theory at both the fixed angle or Gross regime (GR) and the fixed momentum transfer or Regge regime (RR). We calculated in details high energy string scattering amplitudes at arbitrary mass levels for both regimes. We discovered infinite linear relations among fixed angle string amplitudes conjectured by Gross in 1988 from decoupling of high energy zero-norm states (ZNS), and infinite recurrence relations among Regge string amplitudes from Kummer function U and Appell function F_1 . In the GR/RR regime, all high energy string amplitudes can be solved by these linear/recurrence relations so that all GR/RR string amplitudes can be expressed in terms of one single GR/RR string amplitude. In addition, we found an interesting link between string amplitudes of the two regimes, and discovered that at each mass level the ratios among fixed angle amplitudes can be extracted from Regge string scattering amplitudes. This result enables us to argue that the known $SL(5, C)$ dynamical symmetry of the Appell function F_1 is crucial to probe high energy spacetime symmetry of string theory.

Keywords: Symmetries of strings, Hard string scattering amplitudes, Regge string scattering amplitudes, High energy limits, Zero norm states, Linear relations, Recurrence relations

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. INTRODUCTION AND OVERVIEW

One of the fundamental issues of string theory is its spacetime symmetry structure. It has long been believed that string theory consists of huge hidden symmetries. This is strongly suggested by the UV finiteness of quantum string theory, which contains no free parameter and an infinite number of states. On the other hand, the high energy, fixed angle behavior of string scattering amplitudes was known to be very soft exponential fall-off, while that of a local quantum field theory was power law. Presumably, it is these huge hidden symmetries

which soften the UV structure of quantum string theory. In a local quantum field theory, a symmetry principle was postulated, which can be used to determine the interaction of the theory. In string theory, on the contrary, it is the interaction, prescribed by the very tight quantum consistency conditions due to the extendedness of string, which determines the form of the symmetry.

Historically, the first key progress to understand symmetry of string theory was to study, instead of low energy field theory limit, the high energy, fixed angle behavior of hard string scattering (HSS) amplitudes [1–5]. This was motivated by the spontaneously broken symmetries in gauge field theories which were hidden at low energy, but became evident in the high energy behavior of the theory. There were two main conjectures of Gross’s [3, 4] pioneer work in 1988 on this subject. The first one was the existence of an infinite number of linear relations among the scattering amplitudes of different string states that were valid order by order in string perturbation theory at high energies, fixed angle regime or Gross regime (GR). The second was that this symmetry was so powerful as to determine the scattering amplitudes of all the infinite number of string states in terms of a single dilaton (tachyon for the case of open string) scattering amplitudes. However, the symmetry charges of his proposed stringy symmetries were not understood and the proportionality constants or ratios among scattering amplitudes of different string states were not calculated.

The second key to uncover the fundamental symmetry of string theory was zero norm states (ZNS) in the old covariant first quantized (OCFQ) string spectrum. It was proposed that [6–8] spacetime symmetry charges of string theory originate from an infinite number of ZNS with arbitrary high spin in the spectrum. In the context of σ -model approach of string theory, one turns on background fields on the worldsheet energy momentum tensor T . Conformal invariance of the worldsheet then requires, in addition to $D = 26$, cancellation of various q -number anomalies and results to equations of motion of the background fields [9]. It was then shown that [6] for each *spacetime* ZNS, one can systematically construct a *worldsheet* $(1, 1)$ primary field δT_Φ such that

$$T_\Phi + \delta T_\Phi = T_{\Phi+\delta\Phi} \tag{0.1}$$

is satisfied to some order of weak field approximation in the σ -model background fields β function calculation. In the above equation, T_Φ is the worldsheet energy momentum tensor with background fields Φ and $T_{\Phi+\delta\Phi}$ is the new energy momentum tensor with new

background fields $\Phi + \delta\Phi$. Thus for each ZNS one can construct a spacetime symmetry transformation for string background fields.

In addition to the positive norm physical propagating states, there are two types of physical ZNS in the old covariant first quantized open bosonic string spectrum: [9]

$$\text{Type I : } L_{-1} |x\rangle, \text{ where } L_1 |x\rangle = L_2 |x\rangle = 0, L_0 |x\rangle = 0; \quad (0.2)$$

$$\text{Type II : } (L_{-2} + \frac{3}{2}L_{-1}^2) |\tilde{x}\rangle, \text{ where } L_1 |\tilde{x}\rangle = L_2 |\tilde{x}\rangle = 0, (L_0 + 1) |\tilde{x}\rangle = 0. \quad (0.3)$$

While type I states have zero-norm at any spacetime dimension, type II states have zero-norm *only* at $D = 26$. For example, among other stringy symmetries, an inter-particle symmetry transformation for two propagating states at mass level $M^2 = 4$ of open bosonic string can be generated [6]

$$\delta C_{(\mu\nu\lambda)} = \frac{1}{2}\partial_{(\mu}\partial_{\nu}\theta_{\lambda)}^2 - 2\eta_{(\mu\nu}\theta_{\lambda)}^2, \delta C_{[\mu\nu]} = 9\partial_{[\mu}\theta_{\nu]}^2, \quad (0.4)$$

where $\partial^\mu\theta_\mu^2 = 0, (\partial^2 - 4)\theta_\mu^2 = 0$ which are the on-shell conditions of the D_2 vector ZNS with polarization θ_μ^2 [6]

$$|D_2\rangle = [(\frac{1}{2}k_\mu k_\nu \theta_\lambda^2 + 2\eta_{\mu\nu}\theta_\lambda^2)\alpha_{-1}^\mu\alpha_{-1}^\nu\alpha_{-1}^\lambda + 9k_\mu\theta_\nu^2\alpha_{-2}^{[\mu}\alpha_{-1}^{\nu]} - 6\theta_\mu^2\alpha_{-3}^\mu] |0, k\rangle, \quad k \cdot \theta^2 = 0, \quad (0.5)$$

and $C_{(\mu\nu\lambda)}$ and $C_{[\mu\nu]}$ are the background fields of the symmetric spin-three and antisymmetric spin-two propagating states respectively.

In the even higher mass levels, $M^2 = 6$ for example, a new phenomenon begins to show up. There are ambiguities in defining positive-norm spin-two and scalar states due to the existence of ZNS in the same Young representations [8]. As a result, the degenerate spin two and scalar positive-norm states can be gauged to the higher rank fields, the symmetric spin four $D_{\mu\nu\alpha\beta}$ and mixed-symmetric spin three $D_{\mu\nu\alpha}$ in the first order weak field approximation. In fact, for instance, it can be shown [10] that the scattering amplitude involving the positive-norm spin-two state can be expressed in terms of those of spin-four and mixed-symmetric spin-three states due to the existence of a *degenerate* type I and a type II spin-two ZNS. This stringy phenomenon seems to persist to higher mass levels.

This calculation is consistent with the result in the HSS limit. In fact, it can be shown that in the HSS limit all the scattering amplitudes of leading order in energy at each fixed mass level can be expressed in terms of that of the leading trajectory string state with transverse

polarizations on the scattering plane. See Eq.(0.9), Eq.(0.21) and Eq.(0.29) below. One can also justify this decoupling by WSFT to be discussed in section I.D. Finally one expects this decoupling to persist even if one includes the higher order corrections in weak field approximation, as there will be even stronger relations between background fields order by order through iteration.

The calculation of Eq.(0.4) was done in the first order weak field approximation but valid to all energies or all orders in α' . A second order weak field calculation implies an even more interesting spontaneously broken inter-mass level symmetry in string theory [11, 12]. Some implication of the corresponding stringy Ward identity on the scattering amplitudes were discussed in [11, 13] and will be presented in Eq.(0.12). It was then realized that [14, 15] the symmetry in Eq.(0.4) can be reproduced from gauge transformation of Witten string field theory (WSFT) [16] after imposing the no ghost conditions. It is important to note that this stringy symmetry exists only for $D = 26$ thanks to type II ZNS in the OCFQ string spectrum, which is zero norm only when $D = 26$.

Incidentally, it was well known in $2D$ string theory that the operator products of the discrete positive norm states $\psi_{J,M}^+$ form a w_∞ algebra [17–19]

$$\int \frac{dz}{2\pi i} \psi_{J_1, M_1}^+ \psi_{J_2, M_2}^+ = (J_2 M_1 - J_1 M_2) \psi_{J_1+J_2-1, M_1+M_2}^+. \quad (0.6)$$

This is in parallel with the work of Ref [20, 21] where the ground ring structure of ghost number zero operators was identified in the BRST quantization. Interestingly, a set of discrete ZNS $G_{J,M}^+$ with Polyakov momenta can be constructed (see Eq.(3.26) in section III.A.2) and were also shown [22, 23] to carry the spacetime ω_∞ symmetry [17–19] charges of $2D$ string theory [22, 23]

$$\int \frac{dz}{2\pi i} G_{J_1, M_1}^+(z) G_{J_2, M_2}^+(0) = (J_2 M_1 - J_1 M_2) G_{J_1+J_2-1, M_1+M_2}^+(0). \quad (0.7)$$

The calculation above can be generalized to $2D$ superstring theory [23].

One can also use ZNS to calculate spacetime symmetries of string on compact backgrounds. The existence of soliton ZNS at some moduli points was shown to be responsible for the enhanced Kac-Moody symmetry of closed string theory. As a simple example, for the case of $26D$ bosonic closed string compactified on a 2-dimensional torus $T^2 \equiv \frac{R^2}{2\pi\Lambda^2}$, it was found that massless ZNS (including soliton ZNS) form a representation of enhanced

Kac-Moody $SU(3)_R \otimes SU(3)_L$ symmetry at the moduli point (see section IV.A.2)

$$R_1 = R_2 = \sqrt{2}, B = \frac{1}{2}, \vec{e}_1 = (\sqrt{2}, 0), \vec{e}_2 = \left(-\sqrt{\frac{1}{2}}, \sqrt{\frac{3}{2}}\right) \quad (0.8)$$

where Λ^2 is a 2-dimensional lattice with a basis $\left\{R_1 \frac{\vec{e}_1}{\sqrt{2}}, R_2 \frac{\vec{e}_2}{\sqrt{2}}\right\}$, and B is the antisymmetric tensor $B_{ij} = B\epsilon_{ij}$. In this calculation one has four moduli parameters R_1, R_2, B and $\vec{e}_1 \cdot \vec{e}_2$ with $|\vec{e}_i|^2 = 2$. Moreover, an infinite number of massive soliton ZNS at any higher massive level of the spectrum were constructed in [24]. Presumably, these massive soliton ZNS are responsible for enhanced stringy symmetries of the theory.

For the case of open string compactification, unlike the closed string case discussed above, it was found that [25] the soliton ZNS exist only at massive levels. These Chan-Paton soliton ZNS correspond to the existence of enhanced massive stringy symmetries with transformation parameters containing both Einstein and Yang-Mills indices in the case of Heterotic string [11]. In the T-dual picture, these symmetries exist only at some discrete values of compactified radii when N D -branes are coincident [25].

All the above results which are valid to all energies will constitute the part I of this review paper. On the other hand, in part II of this review, we will show that the high energy limit of the discrete ZNS $G_{J,M}^+$ in $2D$ string theory constructed in Eq.(0.7) in part I approaches $\psi_{J,M}^+$ in Eq.(0.6) and thus form a high energy w_∞ symmetry of $2D$ string. This result strongly suggests that the linear relations obtained from decoupling of ZNS in $26D$ string theory are indeed related to the hidden symmetry also for the $26D$ string theory.

In part II of this paper, we will review high energy, fixed angle calculations of HSS amplitudes. The high energy, fixed angle Ward identities derived from the decoupling of ZNS in the HSS limit, which combines the previous two key ideas of probing stringy symmetry, were used to explicitly prove Gross's two conjectures [26–31]. An infinite number of linear relations among high energy scattering amplitudes of different string states were derived. Remarkably, these linear relations were just good enough to fix the proportionality constants or ratios among high energy scattering amplitudes of different string states algebraically at each fixed mass level. The first example calculated was the ratios among HSS amplitudes at mass level $M^2 = 4$ [26, 28] (see the definition of polarizations e^T and e^L after Eq.(0.16) below)

$$\mathcal{T}_{TTT} : \mathcal{T}_{LLT} : \mathcal{T}_{(LT)} : \mathcal{T}_{[LT]} = 8 : 1 : -1 : -1 \quad (0.9)$$

which corresponds to stringy symmetries in the σ -model calculation discussed from Eq.(0.1) to Eq.(0.5). Eq.(0.9) is presumably valid order by order in string perturbation theory as we expect the decoupling of ZNS is valid even for string loop amplitudes [32].

To calculate Eq.(0.9), we note that there are four ZNS at mass level $M^2 = 4$. For type I ZNS, there is one symmetric spin two tensor, one vector and one scalar ZNS. In addition, there is only one vector type II ZNS. The corresponding Ward identities for these four ZNS were calculated to be [13]

$$k_\mu \theta_{\nu\lambda} \mathcal{T}_\chi^{(\mu\nu\lambda)} + 2\theta_{\mu\nu} \mathcal{T}_\chi^{(\mu\nu)} = 0, \quad (0.10)$$

$$\left(\frac{5}{2}k_\mu k_\nu \theta'_\lambda + \eta_{\mu\nu} \theta'_\lambda\right) \mathcal{T}_\chi^{(\mu\nu\lambda)} + 9k_\mu \theta'_\nu \mathcal{T}_\chi^{(\mu\nu)} + 6\theta'_\mu \mathcal{T}_\chi^\mu = 0, \quad (0.11)$$

$$\left(\frac{1}{2}k_\mu k_\nu \theta_\lambda + 2\eta_{\mu\nu} \theta_\lambda\right) \mathcal{T}_\chi^{(\mu\nu\lambda)} + 9k_\mu \theta_\nu \mathcal{T}_\chi^{[\mu\nu]} - 6\theta_\mu \mathcal{T}_\chi^\mu = 0, \quad (0.12)$$

$$\left(\frac{17}{4}k_\mu k_\nu k_\lambda + \frac{9}{2}\eta_{\mu\nu} k_\lambda\right) \mathcal{T}_\chi^{(\mu\nu\lambda)} + (9\eta_{\mu\nu} + 21k_\mu k_\nu) \mathcal{T}_\chi^{(\mu\nu)} + 25k_\mu \mathcal{T}_\chi^\mu = 0 \quad (0.13)$$

where $\theta_{\mu\nu}$ is transverse and traceless, and θ'_λ and θ_λ are transverse vectors. \mathcal{T}_χ 's in the above equations are the mass level $M^2 = 4$, χ -th order string-loop amplitudes. In each equation, we have chosen, say, $v_2(k_2)$ to be the vertex operators constructed from ZNS and $k_\mu \equiv k_{2\mu}$. Note that Eq.(0.12) is the inter-particle Ward identity corresponding to D_2 vector ZNS in Eq.(0.5) obtained by antisymmetrizing those terms which contain $\alpha_{-1}^\mu \alpha_{-2}^\nu$ in the original type I and type II vector ZNS [6]. We will use 1 and 2 for the incoming particles and 3 and 4 for the scattered particles. In the Ward identities, 1, 3 and 4 can be any string states and we have omitted their tensor indices for the cases of excited string states.

In the HSS limit, one enjoys many simplifications in the calculation. First, all polarizations of the amplitudes orthogonal to the scattering plane are of subleading order in energy, and one needs only consider polarizations on the scattering plane. Second, to the leading order in energy, $e^P \simeq e^L$ in the HSS calculation. In the end of the calculation, one ends up with the simple linear equations for leading order amplitudes [26, 28]

$$\mathcal{T}_{LLT}^{5 \rightarrow 3} + \mathcal{T}_{(LT)}^3 = 0, \quad (0.14)$$

$$10\mathcal{T}_{LLT}^{5 \rightarrow 3} + \mathcal{T}_{TTT}^3 + 18\mathcal{T}_{(LT)}^3 = 0, \quad (0.15)$$

$$\mathcal{T}_{LLT}^{5 \rightarrow 3} + \mathcal{T}_{TTT}^3 + 9\mathcal{T}_{[LT]}^3 = 0 \quad (0.16)$$

where $e^P = \frac{1}{M_2}(E_2, k_2, 0) = \frac{k_2}{M_2}$ the momentum polarization, $e^L = \frac{1}{M_2}(k_2, E_2, 0)$ the longitudinal polarization and $e^T = (0, 0, 1)$ the transverse polarization are the three polarizations on the scattering plane. In Eq.(0.14) to Eq.(0.16), we have assigned a relative energy power for each amplitude. For each longitudinal L component, the order is E^2 and for each transverse T component, the order is E . This is due to the definitions of e_L and e_T above, where e_L got one energy power more than e_T . By Eq.(0.15), the naive leading order E^5 term of the energy expansion for \mathcal{T}_{LLT} is forced to be zero. As a result, the real leading order term is E^3 . Similar rule applies to \mathcal{T}_{LLT} in Eq.(0.14) and Eq.(0.16). The solution of these three linear relations gives Eq.(0.9). Eq.(0.9) gives the first evidence of Gross conjecture [3, 4] on HSS amplitudes.

A sample calculation of scattering amplitudes for mass level $M^2 = 4$ [28] justified the ratios calculated in Eq.(0.9). Since the proportionality constants in Eq.(0.9) are independent of particles chosen for vertex $v_{1,3,4}$, for simplicity, we will choose them to be tachyons. For the string-tree level $\chi = 1$, with one tensor v_2 and three tachyons $v_{1,3,4}$, all scattering amplitudes of mass level $M_2^2 = 4$ were calculated to be ($s - t$ channel)

$$\mathcal{T}_{TTT} = -8E^9\mathcal{T}(3)\sin^3\phi_{CM}\left[1 + \frac{3}{E^2} + \frac{5}{4E^4} - \frac{5}{4E^6} + O\left(\frac{1}{E^8}\right)\right], \quad (0.17)$$

$$\begin{aligned} \mathcal{T}_{LLT} = & -E^9\mathcal{T}(3)\left[\sin^3\phi_{CM} + (6\sin\phi_{CM}\cos^2\phi_{CM})\frac{1}{E^2} \right. \\ & \left. - \sin\phi_{CM}\left(\frac{11}{2}\sin^2\phi_{CM} - 6\right)\frac{1}{E^4} + O\left(\frac{1}{E^6}\right)\right], \end{aligned} \quad (0.18)$$

$$\begin{aligned} \mathcal{T}_{[LT]} = & E^9\mathcal{T}(3)\left[\sin^3\phi_{CM} - (2\sin\phi_{CM}\cos^2\phi_{CM})\frac{1}{E^2} \right. \\ & \left. + \sin\phi_{CM}\left(\frac{3}{2}\sin^2\phi_{CM} - 2\right)\frac{1}{E^4} + O\left(\frac{1}{E^6}\right)\right], \end{aligned} \quad (0.19)$$

$$\begin{aligned} \mathcal{T}_{(LT)} = & E^9\mathcal{T}(3)\left[\sin^3\phi_{CM} + \sin\phi_{CM}\left(\frac{3}{2} - 10\cos\phi_{CM} \right. \right. \\ & \left. \left. - \frac{3}{2}\cos^2\phi_{CM}\right)\frac{1}{E^2} - \sin\phi_{CM}\left(\frac{1}{4} + 10\cos\phi_{CM} + \frac{3}{4}\cos^2\phi_{CM}\right)\frac{1}{E^4} + O\left(\frac{1}{E^6}\right)\right] \end{aligned} \quad (0.20)$$

where $\mathcal{T}(N) = \sqrt{\pi}(-1)^{N-1}2^{-N}E^{-1-2N}(\sin\frac{\phi_{CM}}{2})^{-3}(\cos\frac{\phi_{CM}}{2})^{5-2N}\exp\left(-\frac{s\ln s + t\ln t - (s+t)\ln(s+t)}{2}\right)$ is the high energy limit of $\frac{\Gamma(-\frac{s}{2}-1)\Gamma(-\frac{t}{2}-1)}{\Gamma(\frac{u}{2}+2)}$ with $s + t + u = 2N - 8$. We thus have justified Eq.(0.9) with $\mathcal{T}_{TTT}^3 = -8E^9\mathcal{T}(3)\sin^3\phi_{CM}$.

The calculations based on ZNS thus relate [15] gauge transformation of WSFT to high energy string symmetries of Gross. However, in the sample calculation of [5], two of the four high energy amplitudes in Eq.(0.9) were missing, and thus the decoupling of ZNS or unitarity was violated. This is of course due to the unawareness of the importance of ZNS in the saddle-point calculation of [1–5].

The calculations for $M^2 = 4$ above can be generalized to $M^2 = 6$ [28]. To the leading order in energy, one ended up with 8 equations and 9 amplitudes. A calculation showed that [28]

$$\begin{aligned} \mathcal{T}_{TTTT}^4 : \mathcal{T}_{TTLL}^4 : \mathcal{T}_{LLLL}^4 : \mathcal{T}_{TTL}^4 : \mathcal{T}_{LLL}^4 : \tilde{\mathcal{T}}_{LT,T}^4 : \tilde{\mathcal{T}}_{LP,P}^4 : \mathcal{T}_{LL}^4 : \tilde{\mathcal{T}}_{LL}^4 = \\ 16 : \frac{4}{3} : \frac{1}{3} : -\frac{4\sqrt{6}}{9} : -\frac{\sqrt{6}}{9} : -\frac{2\sqrt{6}}{3} : 0 : \frac{2}{3} : 0. \end{aligned} \quad (0.21)$$

A sample calculation of scattering amplitudes for mass level $M^2 = 6$ [28] justified the ratios above calculated by solving 8 linear relations derived from the decoupling of high energy ZNS in the GR. The ratios for $M^2 = 8$ can be found in Eq.(A.15) in the appendix A.

The results of mass level $M^2 = 4, 6$ and 8 can be generalized to arbitrary higher mass levels. From the calculations of Eq.(0.14) to Eq.(0.16), one first observes that only states of the following form [30, 31]

$$|N, 2m, q\rangle \equiv (\alpha_{-1}^T)^{N-2m-2q} (\alpha_{-1}^L)^{2m} (\alpha_{-2}^L)^q |0, k\rangle \quad (0.22)$$

are of leading order in energy in the HSS limit. The choice of only even power $2m$ in α_{-1}^L is the result of the observation that the naive energy order of the amplitudes will in general drop by even number of energy power as can be seen in Eq.(0.14) to Eq.(0.16). Scattering amplitudes corresponding to states with $(\alpha_{-1}^L)^{2m+1}$ turn out to be of subleading order in energy. Many simplifications occur if we apply Ward identities or decoupling of ZNS only on these high energy states in the HSS limit. First, consider the decoupling of type I high energy ZNS

$$L_{-1}|N-1, 2m-1, q\rangle \simeq M|N, 2m, q\rangle + (2m-1)|N, 2m-2, q+1\rangle \quad (0.23)$$

where many terms are omitted because they are not of the form of the leading order. This implies that

$$\mathcal{T}^{(N, 2m, q)} = -\frac{2m-1}{M} \mathcal{T}^{(N, 2m-2, q+1)}. \quad (0.24)$$

Using this relation repeatedly, we get

$$\mathcal{T}^{(N,2m,q)} = \frac{(2m-1)!!}{(-M)^m} \mathcal{T}^{(N,0,m+q)} \quad (0.25)$$

where the double factorial is defined by $(2m-1)!! = \frac{(2m)!}{2^m m!}$.

Next, consider the decoupling of type II high energy ZNS

$$L_{-2}|N-2, 0, q\rangle \simeq \frac{1}{2}|N, 0, q\rangle + M|N, 0, q+1\rangle. \quad (0.26)$$

Again, irrelevant terms are omitted here. From this we deduce that

$$\mathcal{T}^{(N,0,q+1)} = -\frac{1}{2M} \mathcal{T}^{(N,0,q)}, \quad (0.27)$$

which leads to

$$\mathcal{T}^{(N,0,q)} = \frac{1}{(-2M)^q} \mathcal{T}^{(N,0,0)}. \quad (0.28)$$

Our main result for arbitrary mass levels $M^2 = 2(N-1)$ is an immediate deduction of the above two equations, Eq.(0.25) and Eq.(0.28), [30, 31]

$$\frac{T^{(N,2m,q)}}{T^{(N,0,0)}} = \left(-\frac{1}{M}\right)^{2m+q} \left(\frac{1}{2}\right)^{m+q} (2m-1)!!. \quad (0.29)$$

Exactly the same results can also be obtained by two other calculations, the Virasoro constraint calculation and the saddle-point calculation. Here we review the saddle-point calculation. Since the result in Eq.(0.29) is valid for all string loop order, we need only do saddle-point calculation of the string tree level amplitudes. Without loss of generality, we choose particles 1,3 and 4 to be tachyons, and particle 2 to be of the form of Eq.(0.22). The $t-u$ channel contribution to the stringy amplitude at tree level is

$$\begin{aligned} \mathcal{T}^{(N,2m,q)} &= \int_1^\infty dx x^{(1,2)} (1-x)^{(2,3)} \left[\frac{e^T \cdot k_1}{x} - \frac{e^T \cdot k_3}{1-x} \right]^{N-2m-2q} \\ &\quad \cdot \left[\frac{e^P \cdot k_1}{x} - \frac{e^P \cdot k_3}{1-x} \right]^{2m} \left[-\frac{e^P \cdot k_1}{x^2} - \frac{e^P \cdot k_3}{(1-x)^2} \right]^q \end{aligned} \quad (0.30)$$

where $(1,2) = k_1 \cdot k_2$ etc.

In order to apply the saddle-point method, we rewrite the amplitude above into the following form [30, 31]

$$\mathcal{T}^{(N,2m,q)}(K) = \int_1^\infty dx u(x) e^{-Kf(x)}, \quad (0.31)$$

where

$$K \equiv -(1, 2) \rightarrow \frac{s}{2} \rightarrow 2E^2, \quad (0.32)$$

$$\tau \equiv -\frac{(2, 3)}{(1, 2)} \rightarrow -\frac{t}{s} \rightarrow \sin^2 \frac{\phi}{2}, \quad (0.33)$$

$$f(x) \equiv \ln x - \tau \ln(1 - x), \quad (0.34)$$

$$u(x) \equiv \left[\frac{(1, 2)}{M} \right]^{2m+q} (1 - x)^{-N+2m+2q} (f')^{2m} (f'')^q (-e^T \cdot k_3)^{N-2m-2q}. \quad (0.35)$$

The saddle-point for the integration of moduli, $x = x_0$, is defined by

$$f'(x_0) = 0, \quad (0.36)$$

and we have

$$x_0 = \frac{1}{1 - \tau}, \quad 1 - x_0 = -\frac{\tau}{1 - \tau}, \quad f''(x_0) = (1 - \tau)^3 \tau^{-1}. \quad (0.37)$$

It is easy to see that

$$u(x_0) = u'(x_0) = \dots = u^{(2m-1)}(x_0) = 0, \quad (0.38)$$

and

$$u^{(2m)}(x_0) = \left[\frac{(1, 2)}{M} \right]^{2m+q} (1 - x_0)^{-N+2m+2q} (2m)! (f_0'')^{2m+q} (-e^T \cdot k_3)^{N-2m-2q}. \quad (0.39)$$

With these inputs, one can easily evaluate the Gaussian integral associated with the four-point amplitudes

$$\begin{aligned} & \int_1^\infty dx u(x) e^{-Kf(x)} \\ &= \sqrt{\frac{2\pi}{Kf_0''}} e^{-Kf_0} \left[\frac{u_0^{(2m)}}{2^m m! (f_0'')^m K^m} + O\left(\frac{1}{K^{m+1}}\right) \right] \\ &= \sqrt{\frac{2\pi}{Kf_0''}} e^{-Kf_0} \left[(-1)^{N-q} \frac{2^{N-2m-q} (2m)!}{m! M^{2m+q}} \tau^{-\frac{N}{2}} (1 - \tau)^{\frac{3N}{2}} E^N + O(E^{N-2}) \right]. \end{aligned} \quad (0.40)$$

This result shows explicitly that with one tensor and three tachyons, the energy and angle dependence for the four-point HSS amplitudes only depend on the level N

$$\begin{aligned} \lim_{E \rightarrow \infty} \frac{\mathcal{T}^{(N, 2m, q)}}{\mathcal{T}^{(N, 0, 0)}} &= \frac{(-1)^q (2m)!}{m! (2M)^{2m+q}} \\ &= \left(-\frac{2m-1}{M}\right) \dots \left(-\frac{3}{M}\right) \left(-\frac{1}{M}\right) \left(-\frac{1}{2M}\right)^{m+q}, \end{aligned} \quad (0.41)$$

which is consistent with calculation of decoupling of high energy ZNS obtained in Eq.(0.29).

We conclude that there is only one independent component of high energy scattering amplitude at each fixed mass level. Based on this independent component of high energy scattering amplitude, one can then derive the general formula of high energy scattering amplitude for four arbitrary string states, and express them in terms of that of tachyons. This completes the general proof [26–31] of Gross’s two conjectures on high energy symmetry of string theory stated above.

All the above calculations can be extended to the case of hard superstring scattering amplitudes which will be discussed in chapter XIII of this review. However, it was found that [33] there were new HSS amplitudes for the superstring case. The existence of these new high energy scattering amplitudes of string states with polarizations orthogonal to the scattering plane is due to the worldsheet fermion exchange in the correlation functions. These worldsheet fermion exchanges do not exist in the bosonic string correlation functions and is, presumably, related to the high energy massive spacetime fermionic scattering amplitudes in the R-sector of the theory.

Obviously, these new high energy amplitudes create complications for a full understanding of stringy symmetry. Nevertheless, the claim that there is only one independent high energy scattering amplitude at each fixed mass level of the string spectrum persists in the case of superstring theory, at least, for the NS sector of the theory [33].

Incidentally, it was important to discover [26–29] that the result of saddle-point calculation in Refs [1–5] was inconsistent with high energy stringy Ward identities of ZNS calculation in Refs [26–29]. One simple example was the missing of two of the four amplitudes in Eq.(0.9) as has been pointed out previously. A corrected saddle-point calculation was given in [29], where the missing terms of the calculation in Refs [1–5] were identified to recover the stringy Ward identities.

Indeed, it was found [29] that saddle point calculation in [1–5] is only valid for the tachyon amplitude. In general, the results calculated in [1–5] gives the right energy exponent in the scattering amplitudes, but not the energy power factors in front of the exponential for the cases of the *excited string states*. These energy power factors are subleading terms ignored in [1–5] but they are crucial if one wants to get the linear relations among high energy scattering amplitudes conjectured by Gross.

Interestingly, the inconsistency of the saddle point calculation discussed above for the

excited string states was also pointed out by the authors of [34]. The source of disagreement in their so-called group theoretic approach of stringy symmetries stems from the proper choice of local coordinates for the worldsheet saddle points to describe the behavior of the excited string states at high energy limit. It seems that both the ZNS calculation and the calculation based on group theoretic approach agree with tachyon amplitudes obtained in [1–5] (ignore the possible phase factors in the amplitudes to be discussed in the next few paragraphs), but disagree with amplitudes for other excited string states.

The next interesting issues were the calculation of *closed* string scattering amplitudes and their symmetries in the HSS limit [35]. Historically, the open string four tachyon amplitude in the HSS limit was first calculated in the original paper of Veneziano in 1968. On the other hand, the \mathcal{N} -loop closed HSS amplitudes were calculated by the saddle-point method in [1, 2] in 1988. Both open and closed HSS amplitudes exhibit the very soft exponential fall-off behaviors in contrast to the power law behavior of the scattering amplitudes of quantum field theory.

However, an inconsistency arises if one plugs, for example, the tree level four tachyon open and closed string HSS amplitudes calculated by these authors, into the KLT relation (1986)

$$A_{\text{closed}}^{(4)}(s, t, u) = \sin(\pi k_2 \cdot k_3) A_{\text{open}}^{(4)}(s, t) \bar{A}_{\text{open}}^{(4)}(t, u) \quad (0.42)$$

which is valid for *all* kinematic regimes and for *all* string states. This is due to the phase factor $\sin(\pi k_2 \cdot k_3)$ in the above equation which was missing in the closed string saddle-point calculation in [1, 2]. One clue to see the origin of this inconsistency is to note that the saddle-point $x_0 = \frac{1}{1-\tau}$ identified for the open string calculation in Eq.(0.37) is in the regime $[1, \infty)$. So only saddle point calculation for $\bar{A}_{\text{open}}^{(4)}(t, u)$ is reliable, but not that of $A_{\text{open}}^{(4)}(s, t)$ and neither that of closed string amplitude $A_{\text{closed}}^{(4)}(s, t, u)$ [35] by the KLT relation.

Instead of using saddle-point calculation for the closed HSS amplitudes, the above considerations led the authors of [35] to study the relationship between $A_{\text{open}}^{(4)}(s, t)$ and $\bar{A}_{\text{open}}^{(4)}(t, u)$ for arbitrary string states in the HSS limit. *With the help of the infinite linear relations in Eq.(0.29)*, one needs only calculate relationship between $s - t$ and $t - u$ channel HSS amplitudes for the leading trajectory string states. They ended up with the following result in the HSS limit (2006) [35]

$$A_{\text{open}}^{(4)}(s, t) = \frac{\sin(\pi k_2 \cdot k_4)}{\sin(\pi k_1 k_2)} \bar{A}_{\text{open}}^{(4)}(t, u), \quad (0.43)$$

which is valid for four arbitrary string states. It is now clear that due to the phase factor in the above equation, the saddle-point calculation of $A_{\text{open}}^{(4)}(s, t)$ is not reliable, neither for the closed one $A_{\text{closed}}^{(4)}(s, t, u)$ in view of the KLT relation in Eq.(0.42). One can now use the reliable saddle-point calculation of $\bar{A}_{\text{open}}^{(4)}(t, u)$

$$A_{\text{open}}^{(4-\text{tachyon})}(t, u) \simeq (stu)^{-\frac{3}{2}} \exp\left(-\frac{s \ln s + t \ln t + u \ln u}{2}\right), \quad (0.44)$$

and Eq.(0.43) to calculate $A_{\text{open}}^{(4)}(s, t)$ in the HSS limit. The consistent closed string four-tachyon HSS amplitudes can then be calculated by using the KLT relation in Eq.(0.42) to be [35]

$$A_{\text{closed}}^{(4-\text{tachyon})}(s, t, u) \simeq \frac{\sin(\pi t/2) \sin(\pi u/2)}{\sin(\pi s/2)} (stu)^{-3} \exp\left(-\frac{s \ln s + t \ln t + u \ln u}{4}\right) \quad (0.45)$$

The exponential factor in Eq.(0.44) was first discussed by Veneziano [36]. The result for the high energy closed string four-tachyon amplitude in Eq.(0.45) differs from the one calculated in the literature [1, 2] by an oscillating factor $\frac{\sin(\pi t/2) \sin(\pi u/2)}{\sin(\pi s/2)}$. One notes here that the results of Eqs.(0.45), (0.44) and Eq.(0.43) are consistent with the KLT formula, while the previous calculation in [1, 2] is NOT.

Indeed, one might try to use the saddle-point method to calculate the high energy closed string scattering amplitude. The closed string four-tachyon scattering amplitude is

$$\begin{aligned} A_{\text{closed}}^{(4-\text{tachyon})}(s, t, u) &= \int dx dy \exp\left(\frac{k_1 \cdot k_2}{2} \ln|z| + \frac{k_2 \cdot k_3}{2} \ln|1-z|\right) \\ &\equiv \int dx dy (x^2 + y^2)^{-2} [(1-x)^2 + y^2]^{-2} \exp[-K f(x, y)] \end{aligned} \quad (0.46)$$

where $K = \frac{s}{8}$ and $f(x, y) = \ln(x^2 + y^2) - \tau \ln[(1-x)^2 + y^2]$ with $\tau = -\frac{t}{s}$. One can then calculate the "saddle-point" of $f(x, y)$ to be

$$\nabla f(x, y) \big|_{x_0=\frac{1}{1-\tau}, y_0=0} = 0. \quad (0.47)$$

The HSS limit of the closed string four-tachyon scattering amplitude is then calculated to be

$$A_{\text{closed}}^{(4-\text{tachyon})}(s, t, u) \simeq \frac{2\pi}{K \sqrt{\det \frac{\partial^2 f(x_0, y_0)}{\partial x \partial y}}} \exp[-K f(x_0, y_0)] \simeq (stu)^{-3} \exp\left(-\frac{s \ln s + t \ln t + u \ln u}{4}\right), \quad (0.48)$$

which is consistent with the previous one calculated in the literature [1, 2], but is different from the result in Eq.(0.45). However, one notes that

$$\frac{\partial^2 f(x_0, y_0)}{\partial x^2} = \frac{2(1 - \tau)^3}{\tau} = -\frac{\partial^2 f(x_0, y_0)}{\partial y^2}, \frac{\partial^2 f(x_0, y_0)}{\partial x \partial y} = 0, \quad (0.49)$$

which means that (x_0, y_0) is NOT the local minimum of $f(x, y)$, and one should not trust this saddle-point calculation. There was other evidence pointed out by authors of [35] to support this conclusion. Finally, the ratios of closed HSS amplitudes turned out to be the tensor products of two open string ratios

$$\frac{T(N; 2m, 2m'; q, q')}{T(N; 0, 0; 0, 0)} = \left(-\frac{1}{M_2}\right)^{2(m+m') + q + q'} \left(\frac{1}{2}\right)^{m+m' + q + q'} (2m-1)!!(2m'-1)!! \quad (0.50)$$

The relationship between $s-t$ and $t-u$ channels HSS amplitudes in Eq.(0.43) was later argued to be valid for *all* kinematic regime based on monodromy of integration in string amplitude calculation in 2009 [37]. An explicit proof of Eq.(0.43) for arbitrary four string states and all kinematic regimes was given very recently in [38, 39].

The motivation for the author in [37] to calculate Eq.(0.43) was different from the discussion above which was related to the calculation of hard closed string scattering amplitudes. The motivation in [37] was based on the field theory BCJ relation [40] for Yang-Mills gluon color-stripped scattering amplitudes A which was first pointed out and calculated in 2008 to be

$$sA(k_1, k_2, k_3, k_4) - uA(k_1, k_4, k_2, k_3) = 0. \quad (0.51)$$

Note that for the supersymmetric case, there is no tachyon and the low energy massless limit of Eq.(0.43) reproduces Eq.(0.51).

Recently the mass level dependent of Eq.(0.43) was calculated to be [38, 39]

$$\frac{A_{st}^{(p,r,q)}}{A_{tu}^{(p,r,q)}} = (-1)^N \frac{B(-M_1 M_2 + 1, \frac{M_1 M_2}{2})}{B(\frac{M_1 M_2}{2}, \frac{M_1 M_2}{2})} \simeq \frac{\sin \pi (k_2 \cdot k_4)}{\sin \pi (k_1 \cdot k_2)} \quad (0.52)$$

by taking the *nonrelativistic* limit $|\vec{k}_2| \ll M_S$ of Eq.(0.43). In Eq.(0.52), B was the beta function, and k_1, k_3 and k_4 were taken to be tachyons, and k_2 was the following tensor string state

$$V_2 = (i\partial X^T)^p (i\partial X^L)^r (i\partial X^P)^q e^{ik_2 X} \quad (0.53)$$

where

$$N = p + r + q, \quad M_2^2 = 2(N - 1), \quad N \geq 2. \quad (0.54)$$

The generalization of the four point function relation in Eq.(0.43) to higher point string amplitudes can be found in [37]. It is interesting to see that historically the four point (high energy) string BCJ relations Eq.(0.43) [35] were discovered even earlier than the field theory BCJ relations Eq.(0.51)! [40].

The ratios calculated in Eq.(0.50) persist for the case of closed string D-particle scatterings in the HSS limit. For the simple case of $m = 0 = m'$, the ratios were first calculated to be $(-\frac{1}{2M})^{q+q'}$ [41]. The complete ratios were then calculated through a correspondence between HSS ratios and RSS ratios to be discussed in Eq.(0.76) below, and were found to be *factorized* [42] (see section XIV.C)

$$\frac{T_{SD}^{(N;2m,2m';q,q')}}{T_{SD}^{(N;0,0;0,0)}} = \left(-\frac{1}{M_2}\right)^{2(m+m')+q+q'} \left(\frac{1}{2}\right)^{m+m'+q+q'} (2m-1)!!(2m'-1)!! \quad (0.55)$$

It is well known that the closed string-string scattering amplitudes can be factorized into two open string-string scattering amplitudes due to the existence of the KLT formula [43]. On the contrary, there is no physical picture for open string D-particle tree scattering amplitudes and thus no factorization for closed string D-particle scatterings into two channels of open string D-particle scatterings, and hence no KLT-like formula there.

Thus the factorized ratios in HSS regime calculated above came as a surprise. However, these ratios are consistent with the decoupling of high energy ZNS calculated previously in [26–31, 33, 35, 44]. It will be interesting if one can calculate the complete HSS amplitudes directly and see how the *non-factorized amplitudes* can give the result of factorized ratios.

On the other hand, in contrast to the closed string D-particle scatterings in the HSS limit discussed above, it was shown that, instead of the exponential fall-off behavior of the form factors with Regge-pole structure, the HSS amplitudes of closed string scattered from D24-brane, or D-domain-wall, behave as *power-law with Regge-pole structure* [45]. See Eq.(9.72) and Eq.(9.73) in section IX.A.4. This is to be compared with the well-known power law form factors without Regge-pole structure of the D-instanton scatterings.

This discovery makes D-domain-wall scatterings an unique example of a hybrid of string and field theory scatterings. Moreover, it was discovered that [45] the usual linear relations of HSS amplitudes at each fixed mass level, Eq.(0.55), breaks down for the D-domain-wall

scatterings. This result gives a strong evidence that the existence of the infinite linear relations, or stringy symmetries, of HSS amplitudes is responsible for the softer, exponential fall-off HSS scatterings than the power-law field theory scatterings.

Being a consistent theory of quantum gravity, string theory is remarkable for its soft ultraviolet structure. Presumably, this is mainly due to *three* closely related fundamental characteristics of HSS amplitudes. The first is the softer exponential fall-off behavior of the form factors in the HSS in contrast to the power-law field theory scatterings. The second is the existence of infinite Regge poles in the form factor of string scattering amplitudes. The existence of infinite linear relations discussed in part II of the review constitutes the *third* fundamental characteristics of HSS amplitudes.

It will be important to study more string scatterings, which exhibit the above three unusual behaviors in the HSS limit. In section IX.B, we will consider closed string scattered from O-planes. In particular we first calculate massive closed string states at arbitrary mass levels scattered from Orientifold planes in the HSS limit [46]. The scatterings of massless states from Orientifold planes were calculated in the literature by using the boundary states formalism [47–50], and on the worldsheet of real projected plane RP_2 [51]. Many speculations were made about the scatterings of *massive* string states, in particular, for the case of O-domain-wall scatterings. It is one of the purposes of section IV.B to clarify these speculations and to discuss their relations with the three fundamental characteristics of HSS scatterings stated above.

For the generic Op -planes with $p \geq 0$, one expects to get the infinite linear relations except O-domain-wall HSS. For simplicity, we consider only the case of O-particle HSS [46]. For the case of O-particle scatterings, we obtain infinite linear relations among HSS amplitudes of different string states. We also confirm that there exist only t -channel closed string Regge poles in the form factor of the O-particle scatterings amplitudes as expected.

For the case of O-domain-wall scatterings, we find that, like the well-known D-instanton scatterings, the amplitudes behave like field theory scatterings, namely *UV power-law without Regge pole*. In addition, we find that there exist only finite number of t -channel closed string poles in the form factor of O-domain-wall scatterings, and the masses of the poles are bounded by the masses of the external legs [46]. We thus confirm that all massive closed string states do couple to the O-domain-wall as was conjectured previously [51, 52]. This is also consistent with the boundary state descriptions of O-planes.

For both cases of O-particle and O-domain-wall scatterings, we confirm that there exist no s -channel open string Regge poles in the form factor of the amplitudes as O-planes were known to be not dynamical. However, the usual claim that there is a thickness of order $\sqrt{\alpha'}$ for the O-domain-wall is misleading as the UV behavior of its scatterings is power-law instead of exponential fall-off.

In the end of section IX.B, we summarize the Regge pole structures of closed strings states scattered from various D-branes and O-planes in the following table. The s -channel and t -channel scatterings for both D-branes and O-planes are shown in the Fig. 2. For O-plane scatterings, the s -channel open string Regge poles are not allowed since O-planes are not dynamical. For both cases of Domain-wall scatterings, the t -channel closed string Regge poles are not allowed since there is only one kinematic variable instead of two as in the usual cases.

	$p = -1$	$1 \leq p \leq 23$	$p = 24$
D p -branes	X	C+O	O
O p -planes	X	C	X

In this table, "C" and "O" represent infinite Closed string Regge poles and Open string Regge poles respectively. "X" means there are no infinite Regge poles.

In chapter X, following an old suggestion of Mende [53], we calculate high energy massive scattering amplitudes of bosonic string with some coordinates compactified on the torus [54, 55]. We obtain infinite linear relations among high energy scattering amplitudes of different string states in the Hard scattering limit. In addition, we analyze all possible power-law and soft exponential fall-off regimes of high energy compactified bosonic string scatterings by comparing the scatterings with their 26D noncompactified counterparts.

Interestingly, we discover in section X.A the existence of a power-law regime at fixed angle and an exponential fall-off regime at small angle for high energy compactified open string scatterings [55]. These new phenomena never happen in the 26D string scatterings. The linear relations break down as expected in all power-law regimes. The analysis can be extended to the high energy scatterings of the compactified closed string in section X.B, which corrects and extends the results in [54].

At this point, one may ask an important question for the results of Eqs.(0.9), (0.21), (A.15) and (0.29) above, namely, is there any group theoretical structure of the ratios of

these scattering amplitudes? Let's consider a simple analogy from particle physics. The ratios of the nucleon-nucleon scattering processes

$$\begin{aligned} (a) \quad & p + p \rightarrow d + \pi^+, \\ (b) \quad & p + n \rightarrow d + \pi^0, \\ (c) \quad & n + n \rightarrow d + \pi^- \end{aligned} \tag{0.56}$$

can be calculated to be (ignore the tiny mass difference between proton and neutron)

$$T_a : T_b : T_c = 1 : \frac{1}{\sqrt{2}} : 1 \tag{0.57}$$

from $SU(2)$ isospin symmetry. Is there any symmetry structure which can be used to calculate ratios in Eqs.(0.9), (0.21), (A.15) and (0.29)? It turned out that part of the answer can be addressed by studying another high energy regime of string scattering amplitudes, namely, the fixed momentum transfer or Regge regime (RR) [56–65].

In part III of this paper, we will discuss RSS amplitudes and their relations to the fixed angle HSS amplitudes. We will find that the number of RSS amplitudes is much more numerous than that of HSS amplitudes. For example, there are only 4 HSS amplitudes while there are 22 RSS amplitudes at mass level $M^2 = 4$ [63]. This is one of the reason why decoupling of ZNS in the RR, in contrast to the GR, is not good enough to solve RSS amplitudes in terms of one single amplitude at each mass level.

For illustration and to identify the ratios in Eqs.(0.9) from RSS amplitudes, we will first calculate amplitudes at mass level $M^2 = 4$ in the RR

$$s \rightarrow \infty, \sqrt{-t} = \text{fixed (but } \sqrt{-t} \neq \infty). \tag{0.58}$$

The relevant kinematics are

$$e^P \cdot k_1 = -\frac{1}{M_2} \left(\sqrt{p^2 + M_1^2} \sqrt{p^2 + M_2^2} + p^2 \right) \simeq -\frac{s}{2M_2}, \tag{0.59a}$$

$$e^L \cdot k_1 = -\frac{p}{M_2} \left(\sqrt{p^2 + M_1^2} + \sqrt{p^2 + M_2^2} \right) \simeq -\frac{s}{2M_2}, \tag{0.59b}$$

$$e^T \cdot k_1 = 0 \tag{0.59c}$$

and

$$e^P \cdot k_3 = \frac{1}{M_2} \left(\sqrt{q^2 + M_3^2} \sqrt{p^2 + M_2^2} - pq \cos \theta \right) \simeq -\frac{\tilde{t}}{2M_2} \equiv -\frac{t - M_2^2 - M_3^2}{2M_2}, \quad (0.60a)$$

$$e^L \cdot k_3 = \frac{1}{M_2} \left(p \sqrt{q^2 + M_3^2} - q \sqrt{p^2 + M_2^2} \cos \theta \right) \simeq -\frac{\tilde{t}'}{2M_2} \equiv -\frac{t + M_2^2 - M_3^2}{2M_2}, \quad (0.60b)$$

$$e^T \cdot k_3 = -q \sin \phi \simeq -\sqrt{-t}. \quad (0.60c)$$

Note that in contrast to the identification $e^P \simeq e^L$ in the HSS limit, e^P *does not* approach to e^L in the RSS limit.

We will list the relevant RSS amplitudes at mass level $M^2 = 4$ which contain polarizations (e^T, e^L) only. It turned out that there are eight high energy amplitudes in the RR

$$\begin{aligned} & \alpha_{-1}^T \alpha_{-1}^T \alpha_{-1}^T |0\rangle, \alpha_{-1}^L \alpha_{-1}^T \alpha_{-1}^T |0\rangle, \alpha_{-1}^L \alpha_{-1}^L \alpha_{-1}^T |0\rangle, \alpha_{-1}^L \alpha_{-1}^L \alpha_{-1}^L |0\rangle, \\ & \alpha_{-1}^T \alpha_{-2}^T |0\rangle, \alpha_{-1}^T \alpha_{-2}^L |0\rangle, \alpha_{-1}^L \alpha_{-2}^T |0\rangle, \alpha_{-1}^L \alpha_{-2}^L |0\rangle. \end{aligned} \quad (0.61)$$

Among them only four of the above amplitudes are relevant here and can be calculated to be [63]

$$\begin{aligned} A^{TTT} &= \int_0^1 dx \cdot x^{k_1 \cdot k_2} (1-x)^{k_2 \cdot k_3} \cdot \left(\frac{ie^T \cdot k_1}{x} - \frac{ie^T \cdot k_3}{1-x} \right)^3 \\ &\simeq -i (\sqrt{-t})^3 \frac{\Gamma(-\frac{s}{2}-1) \Gamma(-\frac{\tilde{t}}{2}-1)}{\Gamma(\frac{u}{2}+3)} \cdot \left(-\frac{1}{8}s^3 + \frac{1}{2}s \right), \end{aligned} \quad (0.62)$$

$$\begin{aligned} A^{LLT} &= \int_0^1 dx \cdot x^{k_1 \cdot k_2} (1-x)^{k_2 \cdot k_3} \cdot \left(\frac{ie^T \cdot k_1}{x} - \frac{ie^T \cdot k_3}{1-x} \right) \left(\frac{ie^L \cdot k_1}{x} - \frac{ie^L \cdot k_3}{1-x} \right)^2 \\ &\simeq -i (\sqrt{-t}) \left(-\frac{1}{2M_2} \right)^2 \frac{\Gamma(-\frac{s}{2}-1) \Gamma(-\frac{\tilde{t}}{2}-1)}{\Gamma(\frac{u}{2}+3)} \\ &\cdot \left[\left(\frac{1}{4}t - \frac{9}{2} \right) s^3 + \left(\frac{1}{4}t^2 + \frac{7}{2}t \right) s^2 + \frac{(t+6)^2}{2}s \right], \end{aligned} \quad (0.63)$$

$$\begin{aligned} A^{TL} &= \int_0^1 dx \cdot x^{k_1 \cdot k_2} (1-x)^{k_2 \cdot k_3} \cdot \left(\frac{ie^T \cdot k_1}{x} - \frac{ie^T \cdot k_3}{1-x} \right) \left[\frac{e^L \cdot k_1}{x^2} + \frac{e^L \cdot k_3}{(1-x)^2} \right] \\ &\simeq i (\sqrt{-t}) \left(-\frac{1}{2M_2} \right) \frac{\Gamma(-\frac{s}{2}-1) \Gamma(-\frac{\tilde{t}}{2}-1)}{\Gamma(\frac{u}{2}+3)} \\ &\cdot \left[-\left(\frac{1}{8}t + \frac{3}{4} \right) s^3 - \frac{1}{8} (t^2 - 2t) s^2 - \left(\frac{1}{4}t^2 - t - 3 \right) s \right], \end{aligned} \quad (0.64)$$

and

$$\begin{aligned}
A^{LT} &= \int_0^1 dx \cdot x^{k_1 \cdot k_2} (1-x)^{k_2 \cdot k_3} \cdot \left(\frac{ie^L \cdot k_1}{x} - \frac{ie^L \cdot k_3}{1-x} \right) \left[\frac{e^T \cdot k_1}{x^2} + \frac{e^T \cdot k_3}{(1-x)^2} \right] \\
&\simeq i(\sqrt{-t}) \left(-\frac{1}{2M_2} \right) \frac{\Gamma(-\frac{s}{2}-1) \Gamma(-\frac{\tilde{t}}{2}-1)}{\Gamma(\frac{u}{2}+3)} \cdot \left[\frac{3}{4}s^3 - \frac{t}{4}s^2 - \left(\frac{t}{2} + 3 \right) s \right]. \quad (0.65)
\end{aligned}$$

where the kinematic variables (s, t) were used instead of (E, θ) used in the GR. From the above calculation, one can easily see that all the amplitudes are in the same leading order ($\sim s^3$) in the RR. On the other hand, one notes that, for example, the terms $\sqrt{-t}t^2s^2$ in A^{LLT} and A^{TL} are in the leading order in the GR, but are in the subleading order in the RR. On the contrary, the terms $\sqrt{-t}s^3$ in A^{LLT} and A^{TL} are in the subleading order in the GR, but are in the leading order in the RR. These observations suggest that the high energy string scattering amplitudes in the GR and RR contain information complementary to each other.

One important observation for high energy amplitudes in the RR is for those amplitudes with the same structure as those of the GR in Eq.(0.22). The amplitudes A^{TTT} , A^{LLT} , A^{TL} and A^{LT} at mass level $M^2 = 4$ are such examples. For these amplitudes, the relative ratios of the coefficients of the highest power of t in the leading order amplitudes in the RR can be calculated to be [63]

$$A^{TTT} = -i(\sqrt{-t}) \frac{\Gamma(-\frac{s}{2}-1) \Gamma(-\frac{\tilde{t}}{2}-1)}{\Gamma(\frac{u}{2}+3)} \cdot \left(\frac{1}{8}ts^3 \right) \sim \frac{1}{8}, \quad (0.66)$$

$$A^{LLT} = -i(\sqrt{-t}) \left(-\frac{1}{2M_2} \right)^2 \frac{\Gamma(-\frac{s}{2}-1) \Gamma(-\frac{\tilde{t}}{2}-1)}{\Gamma(\frac{u}{2}+3)} \left(\frac{1}{4}ts^3 \right) \sim \frac{1}{64}, \quad (0.67)$$

$$A^{TL} = i(\sqrt{-t}) \left(-\frac{1}{2M_2} \right) \frac{\Gamma(-\frac{s}{2}-1) \Gamma(-\frac{\tilde{t}}{2}-1)}{\Gamma(\frac{u}{2}+3)} \cdot \left(-\frac{1}{8}ts^3 \right) \sim -\frac{1}{32}, \quad (0.68)$$

which reproduces the ratios in the GR in Eq.(0.9). Note that the symmetrized and anti-symmetrized amplitudes are defined as

$$T^{(TL)} = \frac{1}{2} (T^{TL} + T^{LT}), \quad (0.69)$$

$$T^{[TL]} = \frac{1}{2} (T^{TL} - T^{LT}); \quad (0.70)$$

and similarly for the amplitudes $A^{(TL)}$ and $A^{[TL]}$ in the RR. It is interesting to see that $T^{LT} \sim (\alpha_{-1}^L)(\alpha_{-2}^T)|0\rangle$ in the GR is of subleading order in energy, while A^{LT} in the RR is of

leading order in energy. However, the contribution of the amplitude A^{LT} to $A^{(TL)}$ and $A^{[TL]}$ in the RR will not affect the ratios calculated above.

From the calculation above, it was thus believed that there existed intimate link between high energy string scattering amplitudes in the HSS regime and those in the RSS regime. To study this link and to reproduce the ratios in Eq.(0.29) in particular, one was led to calculate RSS amplitudes for arbitrary mass levels. To simplify the calculation, we use the simple kinematics $e^T \cdot k_1 = 0$ in Eq.(0.29) and the energy power counting of the string amplitudes, and end up with the following rules

$$\alpha_{-n}^T : \quad 1 \text{ term (contraction of } ik_3 \cdot X \text{ with } \varepsilon_T \cdot \partial^n X), \quad (0.71)$$

$$\alpha_{-n}^L : \quad \begin{cases} n > 1, & 1 \text{ term} \\ n = 1 & 2 \text{ terms (contraction of } ik_1 \cdot X \text{ and } ik_3 \cdot X \text{ with } \varepsilon_L \cdot \partial^n X). \end{cases} \quad (0.72)$$

A class of the leading order high energy open string states in the RR at each fixed mass level $N = \sum_{n,l>0} np_n + lr_l$ are

$$|p_n, r_l\rangle = \prod_{n>0} (\alpha_{-n}^T)^{p_n} \prod_{l>0} (\alpha_{-l}^L)^{r_l} |0, k\rangle. \quad (0.73)$$

The $s - t$ channel scattering amplitudes of this state with three other tachyonic states can be calculated to be [63]

$$\begin{aligned} A^{(p_n, q_m)} &= \left(-\frac{i}{M_2}\right)^{q_1} U\left(-q_1, \frac{t}{2} + 2 - q_1, \frac{\tilde{t}'}{2}\right) B\left(-1 - \frac{s}{2}, -1 - \frac{t}{2}\right) \\ &\cdot \prod_{n=1} [i\sqrt{-t}(n-1)!]^{p_n} \prod_{m=2} \left[i\tilde{t}'(m-1)! \left(-\frac{1}{2M_2}\right)\right]^{q_m}. \end{aligned} \quad (0.74)$$

In the above, $U(a, c, x)$ is the Kummer function of the second kind. It is crucial to note that $c = \frac{t}{2} + 2 - q_1$, and is not a constant as in the usual case, so U in the above amplitude is not a solution of the Kummer equation. On the contrary, since $a = -q_1$ an integer, the Kummer function in Eq.(11.25) terminated to be a finite sum.

It can be seen from Eq.(0.74) that the RSS amplitudes with spin polarizations corresponding to Eq.(0.22) at each fixed mass level are no longer proportional to each other. The

ratios are t dependent functions and can be calculated to be [63]

$$\frac{A^{(N,2m,q)}(s,t)}{A^{(N,0,0)}(s,t)} = (-1)^m \left(-\frac{1}{2M_2}\right)^{2m+q} (\tilde{t}' - 2N)^{-m-q} (\tilde{t}')^{2m+q} \\ \times \sum_{j=0}^{2m} (-2m)_j \left(-1 + N - \frac{\tilde{t}'}{2}\right)_j \frac{(-2/\tilde{t}')^j}{j!} + O\left\{\left(\frac{1}{\tilde{t}'}\right)^{m+1}\right\}, \quad (0.75)$$

where $(x)_j = x(x+1)(x+2)\cdots(x+j-1)$ is the Pochhammer symbol.

To deduce the link and ensure the following identification for the general mass levels

$$\lim_{\tilde{t}' \rightarrow \infty} \frac{A^{(N,2m,q)}}{A^{(N,0,0)}} = \frac{T^{(N,2m,q)}}{T^{(N,0,0)}} = \left(-\frac{1}{M_2}\right)^{2m+q} \left(\frac{1}{2}\right)^{m+q} (2m-1)!! \quad (0.76)$$

suggested by the explicit calculation for the mass level $M_2^2 = 4$ [63], one needs the following identity

$$\sum_{j=0}^{2m} (-2m)_j \left(-L - \frac{\tilde{t}'}{2}\right)_j \frac{(-2/\tilde{t}')^j}{j!} \\ = 0 \cdot (-\tilde{t}')^0 + 0 \cdot (-\tilde{t}')^{-1} + \cdots + 0 \cdot (-\tilde{t}')^{-m+1} + \frac{(2m)!}{m!} (-\tilde{t}')^{-m} + O\left\{\left(\frac{1}{\tilde{t}'}\right)^{m+1}\right\} \quad (0.77)$$

where $L = 1 - N$ and is an integer. The identity was proved to be valid for any non-negative integer m and any *real* number L by using technique of combinatorial number theory [66]. It was remarkable to first predict [63] the mathematical identity above provided by string theory, and then a rigorous mathematical proof followed [66]. It was also interesting to see that the validity of the above identity includes non-integer values of L which were later shown to be realized by Regge string scatterings in compact space [67]. We thus have shown that the ratios among HSS amplitudes calculated in Eqs.(0.9) and (0.29) can be deduced and extracted from Kummer functions [63, 68, 69]

$$\frac{T^{(N,2m,q)}}{T^{(N,0,0)}} = \lim_{t \rightarrow \infty} \frac{A^{(N,2m,q)}}{A^{(N,0,0)}} = \left(-\frac{1}{2M}\right)^{2m+q} 2^{2m} \lim_{t \rightarrow \infty} (-t)^{-m} U\left(-2m, \frac{t}{2} + 2 - 2m, \frac{t}{2}\right). \quad (0.78)$$

All the above calculations so far can be generalized to four classes of superstring Regge scattering amplitudes [64]. See the discussion in chapter XII.

The next interesting issue is to study relations among RSS amplitudes of different string states. To achieve this, one considers the more general RSS amplitudes corresponding to

three tachyons and one leading order high energy open string states in the RR at each fixed mass level $N = \sum_{n,m,l>0} np_n + mq_m + lr_l$

$$|p_n, q_m, r_l\rangle = \prod_{n>0} (\alpha_{-n}^T)^{p_n} \prod_{m>0} (\alpha_{-m}^P)^{q_m} \prod_{l>0} (\alpha_{-l}^L)^{r_l} |0, k\rangle. \quad (0.79)$$

The $s - t$ channel scattering amplitudes of this state with three other tachyonic states can be calculated to be

$$A^{(p_n; q_m; r_l)} = \int_0^1 dx x^{k_1 \cdot k_2} (1-x)^{k_2 \cdot k_3} \cdot \left[\frac{e^P \cdot k_1}{x} - \frac{e^P \cdot k_3}{1-x} \right]^{q_1} \left[\frac{e^L \cdot k_1}{x} + \frac{e^L \cdot k_3}{1-x} \right]^{r_1} \\ \cdot \prod_{n=1} \left[\frac{(n-1)! e^T \cdot k_3}{(1-x)^n} \right]^{p_n} \prod_{m=2} \left[\frac{(m-1)! e^P \cdot k_3}{(1-x)^m} \right]^{q_m} \prod_{l=2} \left[\frac{(l-1)! e^L \cdot k_3}{(1-x)^l} \right]^{r_l}. \quad (0.80)$$

Finally, the amplitudes can be written as two equivalent expressions [70]

$$A^{(p_n; q_m; r_l)} = \prod_{n>0} [(n-1)! \sqrt{-t}]^{p_n} \cdot \prod_{m>0} \left[-(m-1)! \frac{\tilde{t}}{2M_2} \right]^{q_m} \cdot \prod_{l>1} \left[(l-1)! \frac{\tilde{t}'}{2M_2} \right]^{r_l} \\ \cdot B\left(-\frac{s}{2} - 1, -\frac{t}{2} + 1\right) \left(\frac{1}{M_2}\right)^{r_1} \\ \cdot \sum_{i=0}^{q_1} \binom{q_1}{i} \left(\frac{2}{\tilde{t}}\right)^i \left(-\frac{t}{2} - 1\right)_i U\left(-r_1, \frac{t}{2} + 2 - i - r_1, \frac{\tilde{t}'}{2}\right) \quad (0.81)$$

$$= \prod_{n>0} [(n-1)! \sqrt{-t}]^{p_n} \cdot \prod_{m>1} \left[-(m-1)! \frac{\tilde{t}}{2M} \right]^{q_m} \cdot \prod_{l>0} \left[(l-1)! \frac{\tilde{t}'}{2M} \right]^{r_l} \\ \cdot B\left(-\frac{s}{2} - 1, -\frac{t}{2} + 1\right) \left(-\frac{1}{M_2}\right)^{q_1} \\ \cdot \sum_{j=0}^{r_1} \binom{r_1}{j} \left(\frac{2}{\tilde{t}'}\right)^j \left(-\frac{t}{2} - 1\right)_j U\left(-q_1, \frac{t}{2} + 2 - j - q_1, \frac{\tilde{t}}{2}\right). \quad (0.82)$$

It is easy to see that, for $q_1 = 0$ or $r_1 = 0$, the RSS amplitudes can be expressed in terms of only one single Kummer function $U\left(-r_1, \frac{t}{2} + 2 - i - r_1, \frac{\tilde{t}'}{2}\right)$ or $U\left(-q_1, \frac{t}{2} + 2 - j - q_1, \frac{\tilde{t}}{2}\right)$. In general the RSS amplitudes can be expressed in terms of a finite sum of Kummer functions. One can then solve these Kummer functions at each mass level and express them in terms of RSS amplitudes. Recurrence relations of Kummer functions can then be used to derive recurrence relations among RSS amplitudes [70]. As an example at mass level $M^2 = 4$, the recurrence relation

$$U\left(-3, \frac{t}{2} - 1, \frac{t}{2} - 1\right) + \left(\frac{t}{2} + 1\right) U\left(-2, \frac{t}{2} - 1, \frac{t}{2} - 1\right) - \left(\frac{t}{2} - 1\right) U\left(-2, \frac{t}{2}, \frac{t}{2} - 1\right) = 0 \quad (0.83)$$

leads to the following recurrence relation among Regge string scattering amplitudes

$$M\sqrt{-t}A^{PPP} - 4A^{PPT} + M\sqrt{-t}A^{PPL} = 0. \quad (0.84)$$

In addition, the addition theorem of Kummer function [71]

$$U(a, c, x + y) = \sum_{k=0}^{\infty} \frac{1}{k!} (a)_k (-1)^k y^k U(a + k, c + k, x) \quad (0.85)$$

which terminates to a finite sum for a non-positive integer a can be used to derive inter-mass level recurrence relation of RSS amplitudes. By taking, for example, $a = -1, c = \frac{t}{2} + 1, x = \frac{t}{2} - 1$ and $y = 1$, the theorem gives

$$U\left(-1, \frac{t}{2} + 1, \frac{t}{2}\right) - U\left(-1, \frac{t}{2} + 1, \frac{t}{2} - 1\right) - U\left(0, \frac{t}{2} + 2, \frac{t}{2} - 1\right) = 0. \quad (0.86)$$

Note that the last arguments of Kummer functions in the above equation can be different. It leads to an inter-mass level recurrence relation of RSS amplitudes [70]

$$M(2)(t + 6)A_2^{TP} - 2M(4)^2\sqrt{-t}A_4^{LP} + 2M(4)A_4^{LT} = 0 \quad (0.87)$$

where masses $M(2) = \sqrt{2}, M(4) = \sqrt{4} = 2$, and A_2, A_4 are RSS amplitudes for mass levels $M^2 = 2, 4$ respectively. In deriving Eq.(0.87), it is important to use the fact that the Regge power law behavior for each RSS amplitude in Eq.(0.87) is universal and is mass level independent [63].

Finally, Kummer recurrence relations can also be used to explicitly prove Regge stringy Ward identities or decoupling of ZNS in the RR, but not vice-versa. Thus in the RR, recurrence relations are more fundamental than linear relations derived from decoupling of Regge ZNS. However, only Ward identities derived from the decoupling of Regge ZNS can be generalized to the string loop amplitudes. As an example, it can be shown that, in the Regge limit, the decoupling of the scalar type I Regge ZNS [70]

$$[25(\alpha_{-1}^P)^3 + 9\alpha_{-1}^P(\alpha_{-1}^L)^2 + 9\alpha_{-1}^P(\alpha_{-1}^T)^2 - 9\alpha_{-2}^L\alpha_{-1}^L - 9\alpha_{-2}^T\alpha_{-1}^T - 75\alpha_{-2}^P\alpha_{-1}^P + 50\alpha_{-3}^P] |0, k\rangle \quad (0.88)$$

can be explicitly demonstrated by using the following recurrence relations of Kummer functions

$$U(a - 1, c, x) - (2a - c + x)U(a, c, x) + a(1 + a - c)U(a + 1, c, x) = 0, \quad (0.89)$$

$$U(a, c, x) - aU(a + 1, c, x) - U(a, c - 1, x) = 0, \quad (0.90)$$

$$(c - a - 1)U(a, c - 1, x) - (x + c - 1)U(a, c, x) + xU(a, c + 1, x) = 0. \quad (0.91)$$

Following the same procedure, one can construct infinite number of recurrence relations among RSS amplitudes at arbitrary mass levels which, in general, are independent of Regge

stringy Ward identities derived from the decoupling of Regge ZNS. However, in contrast to Ward identity derived from the decoupling of Regge ZNS like Eq.(0.88), we have no proof at loop levels for other ward identities derived directly from Kummer function recurrence relations. This is the subtle difference between linear relations obtained in the GR and the recurrence relations calculated in the RR discussed in this review. Recurrence relations of higher spin generalization of the BPST vertex operators [61] can also be constructed in this way [72].

Since in general each RSS amplitude was expressed in terms of more than one Kummer function, it was awkward to derive the complete recurrence relations at arbitrary higher mass levels. More recently [73], it was shown that each $26D$ open bosonic RSS amplitude can be expressed in terms of one *single* Appell function F_1 . In fact, the $s - t$ channel RSS amplitudes with string state in Eq.(0.79) and three tachyons can be calculated as [73]

$$A^{(p_n; q_m; r_l)} = \prod_{n=1} [(n-1)! \sqrt{-t}]^{p_n} \prod_{m=1} \left[-(m-1)! \frac{\tilde{t}}{2M_2} \right]^{q_m} \prod_{l=1} \left[(l-1)! \frac{\tilde{t}'}{2M_2} \right]^{r_l} \cdot F_1 \left(-\frac{t}{2} - 1, -q_1, -r_1, -\frac{s}{2}; \frac{s}{\tilde{t}}, \frac{s}{\tilde{t}'} \right) \cdot B \left(-\frac{t}{2} - 1, -\frac{s}{2} - 1 \right) \quad (0.92)$$

where the Appell function F_1 is one of the four extensions of the hypergeometric function ${}_2F_1$ to two variables and is defined to be

$$F_1(a; b, b'; c; x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a)_{m+n} (b)_m (b')_n}{m! n! (c)_{m+n}} x^m y^n \quad (0.93)$$

where $(a)_n = a \cdot (a+1) \cdots (a+n-1)$ is the rising Pochhammer symbol. Note that when a or $b(b')$ is a non-positive integer, the Appell function truncates to a polynomial. This is the case for the Appell function in the RSS amplitudes calculated above. It is important to keep in mind that the expression in Eq.(0.92) is valid only when s in the arguments of F_1 goes to ∞ .

In contrast to the calculation of a sum of Kummer functions, this result made it easier to derive the complete infinite recurrence relations among RSS amplitudes at arbitrary mass levels, which are conjectured to be related to the known $SL(5, C)$ dynamical symmetry of F_1 [74]. For example, the recurrence relation among RSS amplitudes [73]

$$\sqrt{-t} [A^{(N; q_1, r_1)} + A^{(N; q_1-1, r_1+1)}] - M A^{(N; q_1-1, r_1)} = 0 \quad (0.94)$$

for arbitrary mass levels $M^2 = 2(N-1)$ can be derived from recurrence relations of the Appell functions. Eq.(0.94) is a generalization of Eq.(0.84) to arbitrary mass levels. More

general recurrence relations can be obtained similarly. For example, by taking the leading term of s in the Regge limit, one ends up with the recurrence relation for b_2

$$\begin{aligned}
& cx^2 F_1(a; b_1, b_2; c; x, y) \\
& + [(a - b_1 - b_2 - 1) xy^2 + cx^2 - 2cxy] F_1(a; b_1, b_2 + 1; c; x, y) \\
& - [(a + 1) x^2 y - (a - b_2 - 1) xy^2 - cx^2 + cxy] F_1(a; b_1, b_2 + 2; c; x, y) \\
& - (b_2 + 2) x (x - y) y F_1(a; b_1, b_2 + 3; c; x, y) = 0, \tag{0.95}
\end{aligned}$$

which leads to a recurrence relation for RSS amplitudes at arbitrary mass levels [73]

$$\begin{aligned}
& \tilde{t}'^2 A^{(N; q_1, r_1)} \\
& + [\tilde{t}'^2 + \tilde{t} (t - 2\tilde{t}' - 2q_1 - 2r_1 + 4)] \left(\frac{\tilde{t}'}{\sqrt{-t}} \right) A^{(N; q_1, r_1 + 1)} \\
& + [\tilde{t}'^2 - \tilde{t}' (\tilde{t} + t) + \tilde{t} (t - 2r_1 + 4)] \left(\frac{\tilde{t}'}{\sqrt{-t}} \right)^2 A^{(N; q_1, r_1 + 2)} \\
& - 2(r_1 - 2) (\tilde{t}' - \tilde{t}) \left(\frac{\tilde{t}'}{\sqrt{-t}} \right)^3 A^{(N; q_1, r_1 + 3)} = 0. \tag{0.96}
\end{aligned}$$

More higher recurrence relations which contain general number of $l \geq 3$ Appell functions can be found in [75].

More importantly, one can show [70, 73] that these recurrence relations in the Regge limit can be systematically solved so that all RSS amplitudes can be expressed in terms of one amplitude. All these results seem to dual to high energy symmetries of fixed angle string scattering amplitudes discussed in part II [26–28, 30, 31, 33, 44].

We now proceed to show that the recurrence relations of the Appell function F_1 in the *Regge limit* can be systematically solved so that all RSS amplitudes can be expressed in terms of one amplitude. As the first step, we note that in [70] the RSS amplitudes was expressed in terms of finite sum of Kummer functions. There are two equivalent expressions [70] as was previously shown in Eq.(0.82). It is easy to see that, for $q_1 = 0$ or $r_1 = 0$, the RSS amplitudes can be expressed in terms of only one single Kummer function $U\left(-r_1, \frac{t}{2} + 2 - i - r_1, \frac{\tilde{t}'}{2}\right)$ or $U\left(-q_1, \frac{t}{2} + 2 - j - q_1, \frac{\tilde{t}}{2}\right)$, which are thus related to the Ap-

pell function $F_1\left(-\frac{t}{2}-1; 0, -r_1; \frac{s}{2}; -\frac{s}{t}, -\frac{s}{t'}\right)$ or $F_1\left(-\frac{t}{2}-1; -q_1, 0; \frac{s}{2}; -\frac{s}{t}, -\frac{s}{t'}\right)$ respectively

$$\lim_{s \rightarrow \infty} F_1\left(-\frac{t}{2}-1; 0, -r_1; \frac{s}{2}; -\frac{s}{t}, -\frac{s}{t'}\right) = \left(\frac{2}{\tilde{t}'}\right)^{r_1} U\left(-r_1, \frac{t}{2} + 2 - r_1, \frac{\tilde{t}'}{2}\right), \quad (0.97)$$

$$\lim_{s \rightarrow \infty} F_1\left(-\frac{t}{2}-1; -q_1, 0; \frac{s}{2}; -\frac{s}{t}, -\frac{s}{t'}\right) = \left(\frac{2}{\tilde{t}'}\right)^{q_1} U\left(-q_1, \frac{t}{2} + 2 - q_1, \frac{\tilde{t}'}{2}\right). \quad (0.98)$$

On the other hand, it was shown in [70] that the Kummer functions ratio

$$\frac{U(\alpha, \gamma, z)}{U(0, z, z)} = f(\alpha, \gamma, z), \alpha = 0, -1, -2, -3, \dots \quad (0.99)$$

is determined and $f(\alpha, \gamma, z)$ can be calculated by using recurrence relations of $U(\alpha, \gamma, z)$. Note in addition that $U(0, z, z) = 1$ by explicit calculation. We thus conclude that in the Regge limit

$$c = \frac{s}{2} \rightarrow \infty; x, y \rightarrow \infty; a, b_1, b_2 \text{ fixed}, \quad (0.100)$$

the Appell functions $F_1(a; 0, b_2; c; x, y)$ and $F_1(a; b_1, 0; c; x, y)$ are determined up to an overall factor by recurrence relations. The next step is to derive the recurrence relation

$$yF_1(a; b_1, b_2; c; x, y) - xF_1(a; b_1 + 1, b_2 - 1; c; x, y) + (x - y)F_1(a; b_1 + 1, b_2; c; x, y) = 0, \quad (0.101)$$

which can be obtained from two of the four Appell recurrence relations among contiguous functions.

We can now show that in the Regge limit all RSS amplitudes can be expressed in terms of one single amplitude. We will use the short notation $F_1(a; b_1, b_2; c; x, y) = F_1(b_1, b_2)$ in the following. For $b_2 = -1$, by using Eq.(0.101) and the known $F_1(b_1, 0)$ and $F_1(0, b_2)$, one can easily show that $F_1(b_1, -1)$ are determined for all $b_1 = -1, -2, -3, \dots$. Similarly, $F_1(b_1, -2)$ are determined for all $b_1 = -1, -2, -3, \dots$ if one uses the result of $F_1(b_1, -1)$ in addition to Eq.(0.101) and the known $F_1(b_1, 0)$ and $F_1(0, b_2)$. This process can be continued and one ends up with the result that $F_1(b_1, b_2)$ are determined for all $b_1, b_2 = -1, -2, -3, \dots$. This completes the proof that the recurrence relations of the Appell function F_1 in the Regge limit in Eq.(0.92) can be systematically solved so that all RSS amplitudes can be expressed in terms of one amplitude.

In a very recent paper [39], it was discovered that the 26D open bosonic string scattering amplitudes (SSA) of three tachyons and one arbitrary string state can be expressed in terms of the D-type Lauricella functions with associated $SL(K + 3; C)$ symmetry. As a result,

SSA and symmetries or relations among SSA of different string states at various limits calculated previously can be rederived. These include the linear relations conjectured by Gross [3, 4]. and proved in [26–31] in the hard scattering limit, the recurrence relations in the Regge scattering limit derived from Eq.(0.92) and the extended recurrence relations in the nonrelativistic scattering limit [38] discovered recently. Moreover, one can calculate new recurrence relations of SSA which are valid for all energies. We expect more interesting developments on these research directions in the near future.

In addition to the high energy string scatterings discussed in this review, there were other related approaches in the literature discussing higher spin dynamics of string theory. String theory includes infinitely many higher spin massive fields with consistent mutual interactions, and can provide useful hints on the dynamics of higher spin field theory. On the other hand, a better understanding of higher spin dynamics could also help our comprehension of string theory. It is widely believed that the tensionless limit of string [76–82] is a theory of higher spin gauge fields. In flat spacetime a non-trivial field theory dynamics of the tensionless limit of string theory seems to be ruled out by the theorem of Coleman and Mandula. However, the assumptions of this theorem are violated by the presence of a non-trivial cosmological constant, and one may expect a consistent interacting field theory of higher spins on curved space time. One of the most important explicit and nontrivial construction of interacting higher spin gauge theory is Vasiliev’ system in AdS space-time.

In [83], the spectrum of Kaluza-Klein descendants of fundamental string excitations on $AdS_5 \times S^5$ was derived and organized at the higher spin long multiplets of the AdS supergroup $SU(2, 2|4)$ with a rich pattern of shortenings at the higher spin enhancement point. Furthermore, in the tensionless limit, the field equations from BRST quantization of string theory provide a direct route toward local field equations for higher-spin gauge fields [84].

Recently, in [85], one parameter families of parity violating Vasiliev theory were formulated that preserve $N = 6$ SUSY in AdS_4 . The theory was suggested to be dual to the vector model limit of the $N = 6$ $U(N)_k \times U(M)_{-k}$ ABJ theory in the limit of large N and k but finite M . Since the ABJ theory is also dual to type IIA string theory in $AdS_4 \times CP^3$ with flat B-field, it was speculated that the Vasiliev theory must therefore be a limit of this string theory. Roughly speaking, the fundamental string of string theory is simply the flux tube string of the non-Abelian bulk Vasiliev theory. The relations between ABJ vector model, Vasiliev theory, and type IIA string theory suggests a bulk–bulk duality between Vasiliev

theory and type IIA string field theory, which suggests a concrete way of embedding Vasiliev theory into string theory. It is interesting to investigate whether—and in what guise—the huge bulk gauge symmetry of Vasiliev’s description survives in the bulk string sigma model description of the same system.

There existed other approaches of stringy symmetries which include other studies of string collisions in the high energy, fixed momentum transfer regime [56–62], the Hagedorn transition at high temperature [86–88], vertex operator algebra for compactified spacetime or on a lattice [89–91], group theoretical approach of string [34, 92].

Another motivation of studying high energy string scattering is to investigate the gravitational effect, such as black hole formation due to high energy string collision, and to understand the nonlocal behavior of string theory. Nevertheless, in [93], it was shown that there is no evidence that the extendedness of strings produces any long-distance nonlocal effects in high energy scattering, and no grounds have been found for string effects interfering with formation of a black hole either.

Part I

Stringy symmetries at all energies

In the first part of this review, we discuss stringy symmetries which were calculated to be valid for all energies. These include stringy symmetries calculated by (1) σ -model approach of string theory in the weak field approximation, (2) decoupling of ZNS and stringy Ward identities, (3) Witten’s string field theory, (4) Discrete ZNS and w_∞ symmetry of $2D$ string and (5) Soliton ZNS and enhanced stringy gauge symmetries. We will concentrate on the idea of ZNS and its applications to various calculations of stringy symmetries.

In chapter I we apply ZNS to Sigma model calculation of stringy symmetries [6–8]. We calculate generalized stringy symmetries of massive background fields [6, 7]. We discover the existence of inter-particle, inter-spin symmetry [6] for higher spin string background fields. In addition, we demonstrate the decoupling of degenerate positive-norm states by using two approaches, the σ -model calculation [8] and Witten’s string field theory [14]. All these results are consistent with calculations of high energy string scattering amplitudes which will be

discussed in details in part II and part III. In chapter II, we give a prescription to simplify the calculation of ZNS for higher mass levels [94]. In chapter III, we calculate [22, 23] a set of $2D$ string ZNS with discrete Polyakov momenta and show that its operator algebra forms the w_∞ symmetry algebra of $2D$ string theory. Incidentally, In chapter V of part II, the corresponding high energy ZNS will be shown to form a high energy w_∞ symmetry [31]. These results strongly suggest that ZNS are symmetry charges of $26D$ string theory. In chapter IV we calculate soliton ZNS in compact spaces for both closed [24] and open string [25] theories and study their relations to enhanced stringy gauge symmetries.

I. ZERO NORM STATES (ZNS) AND SIGMA MODEL CALCULATION OF STRINGY SYMMETRIES

In the first chapter, we review the calculations of string symmetries from ZNS without taking the high energy limit. In the OCFQ spectrum of $26D$ open bosonic string theory, the solutions of physical state conditions include positive-norm propagating states and two types of ZNS. The latter are [9]

$$\text{Type I : } L_{-1} |x\rangle, \text{ where } L_1 |x\rangle = L_2 |x\rangle = 0, L_0 |x\rangle = 0; \quad (1.1)$$

$$\text{Type II : } (L_{-2} + \frac{3}{2}L_{-1}^2) |\tilde{x}\rangle, \text{ where } L_1 |\tilde{x}\rangle = L_2 |\tilde{x}\rangle = 0, (L_0 + 1) |\tilde{x}\rangle = 0. \quad (1.2)$$

While type I states have zero-norm at any spacetime dimension, type II states have zero-norm *only* at $D = 26$. It can be shown [9] that the string spectrum is ghost-free provided that $D = 26$ and the Regge intercept $a = 1$ or $D \leq 25$ and $a \leq 1$. However, there are far more ZNS for the former case than that of the latter case. Thus the choice of $D = 26$ case is closely related to the existence of type II ZNS which is crucial in the discussion of this paper. Eqs.(1.1) and Eqs.(1.2) will be extensively used in the review. Some explicit solutions of ZNS can be found in [6, 94] and will be discussed in chapter II.

In the σ -model approach of string theory, one turns on background fields on the worldsheet energy momentum tensor T . Conformal invariance of the worldsheet then requires, in addition to $D = 26$, cancellation of various q -number anomalies and results to equations of motion of the background fields. A spacetime effective action can then be constructed and used to reproduce string scattering amplitudes. This was a powerful method to study dynamics of the string modes [9]. On the other hand, it was suggested that a spacetime

symmetry transformation $\delta\Phi$ for a background field Φ can be generated by a *worldsheet* generator h [95]

$$T_\Phi + i[h, T_\Phi] = T_{\Phi+\delta\Phi} \quad (1.3)$$

where T_Φ is the worldsheet energy momentum tensor with background fields Φ and $T_{\Phi+\delta\Phi}$ is the new energy momentum tensor with new background fields $\Phi + \delta\Phi$. However, there was no systematic prescription to calculate the worldsheet generator h .

It was then shown that [6] for each *spacetime* ZNS, one can systematically construct a δT_Φ such that

$$T_\Phi + \delta T_\Phi = T_{\Phi+\delta\Phi} \quad (1.4)$$

was satisfied to some order of weak field approximation in the background fields β function calculation. It turned out that Eqs.(1.4) gave the complete symmetry transformations for string modes while Eqs.(1.3) did not. Indeed, there were many symmetry transformations which can not be generated by a worldsheet generator h . One important example was the inter-particle symmetry transformation Eqs.(0.4) generated by the D_2 *type II* ZNS in Eqs.(0.5). In contrast to the usual σ -model loop expansion (or α' expansion) of the string background field calculation¹, which was nonrenormalizable for the massive background fields, it turned out that weak field approximation was the more convenient expansion to deal with massive background fields.

A. Stringy symmetries of massive background fields

In this section, as illustrations, we calculate stringy symmetries for $26D$ open bosonic string up to mass levels $M^2 = 4$. All physical states including positive-norm propagating states and two types of ZNS can be found in chapter II. It was demonstrated in the first order weak field approximation of the string modes that for each ZNS in the OCFQ $26D$ open bosonic string spectrum, there corresponds an on-shell gauge transformation for the

¹ See section 3.4 of [9] and references there.

positive-norm background field ($\alpha' \equiv \frac{1}{2}$)[6, 7] :

$$M^2 = 0 : \quad \delta A_\mu = \partial_\mu \theta; \quad (1.5a)$$

$$\partial^2 \theta = 0. \quad (1.5b)$$

$$M^2 = 2 : \quad \delta B_{\mu\nu} = \partial_{(\mu} \theta_{\nu)}; \quad (1.6a)$$

$$\partial^\mu \theta_\mu = 0, (\partial^2 - 2)\theta_\mu = 0. \quad (1.6b)$$

$$\delta B_{\mu\nu} = \frac{3}{2} \partial_\mu \partial_\nu \theta - \frac{1}{2} \eta_{\mu\nu} \theta; \quad (1.7a)$$

$$(\partial^2 - 2)\theta = 0. \quad (1.7b)$$

$$M^2 = 4 : \quad \delta C_{\mu\nu\lambda} = \partial_{(\mu} \theta_{\nu\lambda)}; \quad (1.8a)$$

$$\partial^\mu \theta_{\mu\nu} = \theta_\mu{}^\mu = 0, (\partial^2 - 4)\theta_{\mu\nu} = 0. \quad (1.8b)$$

$$\delta C_{(\mu\nu\lambda)} = \frac{5}{2} \partial_{(\mu} \partial_\nu \theta_{\lambda)}^1 - \eta_{(\mu\nu} \theta_{\lambda)}^1; \quad (1.9a)$$

$$\partial^\mu \theta_\mu^1 = 0, (\partial^2 - 4)\theta_\mu^1 = 0. \quad (1.9b)$$

$$\delta C_{(\mu\nu\lambda)} = \frac{3}{5} \partial_\mu \partial_\nu \partial_\lambda \theta - \frac{1}{5} \eta_{(\mu\nu} \partial_{\lambda)} \theta; \quad (1.10a)$$

$$(\partial^2 - 4)\theta = 0. \quad (1.10b)$$

In the above equations, A , B , C are positive-norm background fields, θ s represent zero-norm background fields, and $\partial^2 \equiv \partial^\mu \partial_\mu$. There are on-mass-shell, gauge and traceless conditions on the transformation parameters θ s, which will correspond to BRST ghost fields in a one-to-one manner in WSFT [14]. This will be discussed in section I.D. Eqs.(1.5a),

(1.5b) is easily identified to be the on-shell gauge transformation of photon. Note that, for example, Eqs.(1.7a), (1.7b) is the residual on-shell gauge transformation induced by a type II ZNS at mass level $M^2 = 2$.

Similar massive stringy symmetry transformations can be constructed for superstring. In particular, based on ZNS calculations, an infinite number of Heterotic massive symmetry transformations [11] with parameters $\theta_\mu^{(ab)}$, $\theta_{[\mu\nu]}^{(ab)}$ etc. containing both Einstein and $E_8 \otimes E_8$ (or $SO(32)$) Yang-Mills indices can be constructed in the 10D Heterotic string theories [96].

B. Inter-particle stringy symmetries

It is interesting to see that an inter-particle symmetry transformation for two high spin states at mass level $M^2 = 4$ can be generated [6]

$$\delta C_{(\mu\nu\lambda)} = \left(\frac{1}{2}\partial_{(\mu}\partial_{\nu}\theta_{\lambda)}^2 - 2\eta_{(\mu\nu}\theta_{\lambda)}^2\right), \delta C_{[\mu\nu]} = 9\partial_{[\mu}\theta_{\nu]}^2 \quad (1.11)$$

where $\partial^\mu\theta_\mu^2 = 0$, $(\partial^2 - 4)\theta_\mu^2 = 0$ which are the on-shell conditions of the mixed type I and type II D_2 vector ZNS

$$|D_2\rangle = \left[\left(\frac{1}{2}k_\mu k_\nu \theta_\lambda + 2\eta_{\mu\nu}\theta_\lambda^2\right)\alpha_{-1}^\mu \alpha_{-1}^\nu \alpha_{-1}^\lambda + 9k_\mu \theta_\nu^2 \alpha_{-2}^{[\mu} \alpha_{-1}^{\nu]} - 6\theta_\mu^2 \alpha_{-3}^\mu\right] |0, k\rangle, \quad k \cdot \theta^2 = 0, \quad (1.12)$$

and $C_{(\mu\nu\lambda)}$ and $C_{[\mu\nu]}$ are the background fields of the symmetric spin-three and antisymmetric spin-two states respectively at the mass level $M^2 = 4$. It is important to note that the decoupling of the D_2 vector ZNS, or unitarity of the theory, implies simultaneous change of both $C_{(\mu\nu\lambda)}$ and $C_{[\mu\nu]}$, thus they form a gauge multiplet. This is a generic feature for background fields of higher massive levels in the σ -model calculation of string theory. One might want to generalize the calculation to the second order weak background fields to see the inter-mass level symmetry. This however suffers from the so-called non-perturbative non-renormalizability of $2d$ σ -model and one is forced to introduce infinite number of counter-terms to preserve the worldsheet conformal invariance [97, 98].

Note that θ_μ^2 in Eqs.(1.11), (1.12) are some linear combination of the original type I and type II vector ZNS calculated by Eqs.(1.1), (1.2). This inter-particle stringy symmetry is consistent with the linear relations among high energy, fixed angle scattering amplitudes of $C_{(\mu\nu\lambda)}$ and $C_{[\mu\nu]}$, which will be discussed in details in part II of the review.

C. Decoupling of degenerate positive-norm states

In the even higher mass levels, $M^2 = 6$ for example, a new phenomenon begins to show up. Indeed, there are ambiguities in defining positive-norm spin-two and scalar states due to the existence of ZNS in the same Young representations [8]. As a result, the degenerate spin two and scalar positive-norm states can be gauged to the higher rank fields $D_{\mu\nu\alpha\beta}$ and mixed-symmetric $D_{\mu\nu\alpha}$ in the first order weak field approximation. Instead of calculating the stringy gauge symmetry at level $M^2 = 6$, we will only concentrate on the equation of motions. Take the energy-momentum tensor on the worldsheet boundary in the first order weak field approximation to be of the following form.

$$T(\tau) = -\frac{1}{2}\eta_{\mu\nu}\partial_\tau X^\mu\partial_\tau X^\nu + D_{\mu\nu\alpha\beta}\partial_\tau X^\mu\partial_\tau X^\nu\partial_\tau X^\alpha\partial_\tau X^\beta + D_{\mu\nu\alpha}\partial_\tau X^\mu\partial_\tau X^\nu\partial_\tau^2 X^\alpha \\ + D_{\mu\nu}^0\partial_\tau^2 X^\mu\partial_\tau^2 X^\nu + D_{\mu\nu}^1\partial_\tau X^\mu\partial_\tau^3 X^\nu + D_\mu\partial_\tau^4 X^\mu, \quad (1.13)$$

where τ is the worldsheet time, $X \equiv X(\tau)$. This is the most general worldsheet coupling in the generalized σ -model approach consistent with vertex operator consideration [99, 100]. The conditions to cancel all q -number worldsheet conformal anomalous terms correspond to cancelling all kinds of loop divergences [101] up to the four loop order in the $2d$ conformal field theory. It is easier to use $T \cdot T$ operator-product calculation and the conditions read [8]

$$2\partial^\mu D_{\mu\nu\alpha\beta} - D_{(\nu\alpha\beta)} = 0, \quad (1.14a)$$

$$\partial^\mu D_{\mu\nu\alpha} - 2D_{\nu\alpha}^0 - 3D_{\nu\alpha}^1 = 0, \quad (1.14b)$$

$$\partial^\mu D_{\mu\nu}^1 - 12D_\nu = 0, \quad (1.14c)$$

$$3D_{\mu\nu\alpha}^\mu + \partial^\mu D_{\nu\alpha\mu} - 3D_{(\nu\alpha)}^1 = 0, \quad (1.14d)$$

$$D_{\mu\nu}^\mu + 4\partial^\mu D_{\mu\nu}^0 - 24D_\nu = 0, \quad (1.14e)$$

$$2D_{\mu\nu}^\nu + 3\partial^\nu D_{\mu\nu}^1 - 12D_\mu = 0, \quad (1.14f)$$

$$2D_\mu^0{}^\mu + 3D_\mu^1{}^\mu + 12\partial^\mu D_\mu = 0, \quad (1.14g)$$

$$(\partial^2 - 6)\phi = 0. \quad (1.14h)$$

Here, ϕ represents all background fields introduced in Eqs.(1.13). It is now clear through Eq.(1.14b) and Eq.(1.14d) that both $D_{\mu\nu}^0$ and $D_{(\mu\nu)}^1$ can be expressed in terms of $D_{\mu\nu\alpha\beta}$ and

$D_{\mu\nu\alpha} \cdot D_{[\mu\nu]}^1$ can be expressed in terms of $D_{\mu\nu\alpha\beta}$ and $D_{\mu\nu\alpha}$ by Eq.(1.14b).Eq.(1.14a) and Eq.(1.14c) imply that $D_{(\mu\nu\alpha)}$ and D_μ can also be expressed in terms of $D_{\mu\nu\alpha\beta}$ and mixed-symmetric $D_{\mu\nu\alpha}$. Finally Eqs.(1.14e) to (1.14g) are the gauge conditions for $D_{\mu\nu\alpha\beta}$ and mixed-symmetric $D_{\mu\nu\alpha}$ after substituting $D_{\mu\nu}^0$, $D_{\mu\nu}^1$ and D_μ in terms of $D_{\mu\nu\alpha\beta}$ and mixed-symmetric $D_{\mu\nu\alpha}$. The remaining scalar particle has automatically been gauged to higher rank fields since Eq.(1.13) is already the most general form of background-field coupling. This means that the degenerate spin two and scalar positive-norm states can be gauged to the higher rank fields $D_{\mu\nu\alpha\beta}$ and mixed-symmetric $D_{\mu\nu\alpha}$ in the first order weak field approximation.

In fact, for instance, it can be explicitly shown [10] that the scattering amplitude involving the positive-norm spin-two state can be expressed in terms of those of spin-four and mixed-symmetric spin-three states due to the existence of a *degenerate* type I and a type II spin-two ZNS. Although all the four-point amplitudes considered in Ref. [10] contain three tachyons, the argument can be easily generalized to more general amplitudes. This is very different from the analysis of lower massive levels where all positive-norm states seem to have independent scattering amplitudes.

Presumably, this decoupling phenomenon comes from the ambiguity in defining positive-norm states due to the existence of ZNS in the same Young representations. We will justify this decoupling by WSFT in the next section. Finally one expects this decoupling to persist even if one includes the higher order corrections in weak field approximation, as there will be even stronger relations between background fields order by order through iteration.

D. Witten's string field theory (WSFT) calculations

It would be much more convincing if one can rederive the stringy phenomena discussed in the previous sections from WSFT. Not only can one compare the first quantized string with the second quantized string, but also the old covariant quantized string with the BRST quantized string. Although the calculation is lengthy, the result, as we shall see, are still controllable by utilizing the results from first quantized approach in previous sections.

There exist important consistency checks of first quantized string results from WSFT in the literature, e.g. the rederivation of Veneziano and Kubo-Nielson amplitudes from WSFT [102]. In some stringy cases, calculations can only be done in string field theory ap-

proach. For example, the pp-wave string amplitudes can only be calculated in the light-cone string field theory [103]. Therefore, a consistent check by both first and second quantized approaches of any reliable string results would be of great importance.

The infinitesimal gauge transformation of WSFT is

$$\delta\Phi = Q_B\Lambda + g_0(\Phi * \Lambda - \Lambda * \Phi). \quad (1.15)$$

To compare with our first quantized results in previous sections, we only need to calculate the first term on the right hand side of Eq.(1.15). Up to the second massive level, Φ and Λ can be expressed as

$$\begin{aligned} \Phi = & \left\{ \phi(x) + iA_\mu(x)\alpha_{-1}^\mu + \alpha(x)b_{-1}c_0 - B_{\mu\nu}(x)\alpha_{-1}^\mu\alpha_{-1}^\nu + iB_\mu(x)\alpha_{-2}^\mu \right. \\ & + i\beta_\mu(x)\alpha_{-1}^\mu b_{-1}c_0 + \beta^0(x)b_{-2}c_0 + \beta^1(x)b_{-1}c_{-1} \\ & - iC_{\mu\nu\lambda}(x)\alpha_{-1}^\mu\alpha_{-1}^\nu\alpha_{-1}^\lambda - C_{\mu\nu}(x)\alpha_{-2}^\mu\alpha_{-1}^\nu + iC_\mu(x)\alpha_{-3}^\mu \\ & - \gamma_{\mu\nu}(x)\alpha_{-1}^\mu\alpha_{-1}^\nu b_{-1}c_0 + i\gamma_\mu^0(x)\alpha_{-1}^\mu b_{-2}c_0 + i\gamma_\mu^1(x)\alpha_{-1}^\mu b_{-1}c_{-1} + i\gamma_\mu^2(x)\alpha_{-2}^\mu b_{-1}c_0 \\ & \left. + \gamma^0(x)b_{-3}c_0 + \gamma^1(x)b_{-2}c_{-1} + \gamma^2(x)b_{-1}c_{-2} \right\} c_1 |k\rangle, \end{aligned} \quad (1.16)$$

$$\begin{aligned} \Lambda = & \left\{ \epsilon^0(x)b_{-1} - \epsilon_{\mu\nu}^0(x)\alpha_{-1}^\mu\alpha_{-1}^\nu b_{-1} + i\epsilon_\mu^0(x)\alpha_{-1}^\mu b_{-1} + i\epsilon_\mu^1(x)\alpha_{-2}^\mu b_{-1} + i\epsilon_\mu^2(x)\alpha_{-1}^\mu b_{-2} \right. \\ & \left. + \epsilon^1(x)b_{-2} + \epsilon^2(x)b_{-3} + \epsilon^3(x)b_{-1}b_{-2}c_0 \right\} |\Omega\rangle \end{aligned} \quad (1.17)$$

where Φ and Λ are restricted to ghost number 1 and 0 respectively, and the BRST charge is

$$Q_B = \sum_{n=-\infty}^{\infty} L_{-n}^{matt} c_n + \sum_{m,n=-\infty}^{\infty} \frac{m-n}{2} : c_m c_n b_{-m-n} : - c_0. \quad (1.18)$$

The transformation one gets for each mass level are

$$M^2 = 0, \quad \delta A_\mu = \partial_\mu \epsilon^0, \quad (1.19a)$$

$$\delta\alpha = \frac{1}{2}\partial^2 \epsilon^0; \quad (1.19b)$$

$$M^2 = 2, \quad \delta B_{\mu\nu} = -\partial_{(\mu}\epsilon_{\nu)}^0 - \frac{1}{2}\epsilon^1\eta_{\mu\nu}, \quad (1.20a)$$

$$\delta B_\mu = -\partial_\mu\epsilon^1 + \epsilon_\mu^0, \quad (1.20b)$$

$$\delta\beta_\mu = \frac{1}{2}(\partial^2 - 2)\epsilon_\mu^0, \quad (1.20c)$$

$$\delta\beta^0 = \frac{1}{2}(\partial^2 - 2)\epsilon_\mu^1, \quad (1.20d)$$

$$\delta\beta^1 = -\partial^\mu\epsilon_\mu^0 - 3\epsilon^1; \quad (1.20e)$$

$$M^2 = 4, \quad \delta C_{\mu\nu\lambda} = -\partial_{(\mu}\epsilon_{\nu\lambda)}^0 - \frac{1}{2}\epsilon_{(\mu}^2\eta_{\nu\lambda)}, \quad (1.21a)$$

$$\delta C_{[\mu\nu]} = -\partial_{[\nu}\epsilon_{\mu]}^1 - \partial_{[\mu}\epsilon_{\nu]}^2, \quad (1.21b)$$

$$\delta C_{(\mu\nu)} = -\partial_{(\nu}\epsilon_{\mu)}^1 - \partial_{(\mu}\epsilon_{\nu)}^2 + 2\epsilon_{\mu\nu}^0 - \epsilon^2\eta_{\mu\nu}, \quad (1.21c)$$

$$\delta C_\mu = -\partial_\mu\epsilon^2 + 2\epsilon_\mu^1 + \epsilon_\mu^2, \quad (1.21d)$$

$$\delta\gamma_{\mu\nu} = \frac{1}{2}(\partial^2 - 4)\epsilon_{\mu\nu}^0 - \frac{1}{2}\epsilon^3\eta_{\mu\nu}, \quad (1.21e)$$

$$\delta\gamma_\mu^0 = \frac{1}{2}(\partial^2 - 4)\epsilon_\mu^2 + \partial_\mu\epsilon^3, \quad (1.21f)$$

$$\delta\gamma_\mu^1 = -2\partial^\nu\epsilon_{\nu\mu}^0 - 2\epsilon_\mu^1 - 3\epsilon_\mu^2, \quad (1.21g)$$

$$\delta\gamma_\mu^2 = \frac{1}{2}(\partial^2 - 4)\epsilon_\mu^1 - \partial_\mu\epsilon^3, \quad (1.21h)$$

$$\delta\gamma^0 = \frac{1}{2}(\partial^2 - 4)\epsilon^2 - \epsilon^3, \quad (1.21i)$$

$$\delta\gamma^1 = -\partial^\mu\epsilon_\mu^2 - 4\epsilon^2 - 2\epsilon^3, \quad (1.21j)$$

$$\delta\gamma^2 = -2\partial^\mu\epsilon_\mu^1 - 5\epsilon^2 + 4\epsilon^3 + \epsilon_\mu^0{}^\mu. \quad (1.21k)$$

It is interesting to note that Eq.(1.19b) corresponds to the lifting of on-mass-shell condition in eqs Eq.(1.5b). Meanwhile Eq.(1.20c) and Eq.(1.20d) correspond to on-mass-shell condition in Eq.(1.7b) and Eq.(1.6b); Eq.(1.20e) corresponds to the gauge condition in Eq.(1.6b). Similar correspondence applies to level $M^2 = 4$. Eq.(1.21e), Eq.(1.21f), Eq.(1.21h) and Eq.(1.21i) correspond to on-mass-shell conditions in Eq.(1.8b), Eq.(1.9b), Eq.(1.12) and Eq.(1.10b). Eq.(1.21g), Eq.(1.21j) and Eq.(1.21k) correspond to gauge conditions in Eq.(1.8b), Eq.(1.9b) and Eq.(1.12). The traceless condition in Eq.(1.8b) corresponds to the trace part of Eq.(1.21e). Also, only ZNS transformation parameters appear on the r.h.s. of matter transformation A, B, C , and all ghost transformations correspond, in a one-to

one manner, to the lifting of on-shell conditions (including on-mass-shell, gauge and traceless conditions) in the OCFQ approach.

These important observations simplify the demonstration of decoupling of degenerate positive-norm states at higher mass levels, $M^2 = 6$ and $M^2 = 8$ more specifically in WSFT. We will present the calculation for level $M^2 = 6$. The calculation for $M^2 = 8$ was discussed in [14]. For $M^2 = 4$, it can be checked that only $C_{\mu\nu\lambda}$ and $C_{[\mu\nu]}$ are dynamically independent and they form a gauge multiplet, which is consistent with result of first quantized calculation presented in the previous sections .

We now show the decoupling phenomenon for the third massive level $M^2 = 6$, in which Φ and Λ can be expanded as

$$\begin{aligned} \Phi_4 = & \left\{ D_{\mu\nu\alpha\beta}(x)\alpha_{-1}^\mu\alpha_{-1}^\nu\alpha_{-1}^\alpha\alpha_{-1}^\beta - iD_{\mu\nu\alpha}(x)\alpha_{-1}^\mu\alpha_{-1}^\nu\alpha_{-2}^\alpha - D_{\mu\nu}^0(x)\alpha_{-2}^\mu\alpha_{-2}^\nu - D_{\mu\nu}^1(x)\alpha_{-1}^\mu\alpha_{-3}^\nu \right. \\ & + iD_\mu(x)\alpha_{-4}^\mu - i\xi_{\mu\nu\alpha}(x)\alpha_{-1}^\mu\alpha_{-1}^\nu\alpha_{-1}^\alpha b_{-1}c_0 - \xi_{\mu\nu}^0(x)\alpha_{-2}^\mu\alpha_{-1}^\nu b_{-1}c_0 - \xi_{\mu\nu}^1(x)\alpha_{-1}^\mu\alpha_{-1}^\nu b_{-2}c_0 \\ & - \xi_{\mu\nu}^2(x)\alpha_{-1}^\mu\alpha_{-1}^\nu b_{-1}c_{-1} + i\xi_\mu^0(x)\alpha_{-3}^\mu b_{-1}c_0 + i\xi_\mu^1(x)\alpha_{-2}^\mu b_{-2}c_0 + i\xi_\mu^2(x)\alpha_{-1}^\mu b_{-3}c_0 \\ & + i\xi_\mu^3(x)\alpha_{-2}^\mu b_{-1}c_{-1} + i\xi_\mu^4(x)\alpha_{-1}^\mu b_{-2}c_{-1} + i\xi_\mu^5(x)\alpha_{-1}^\mu b_{-1}c_{-2} + \xi^0(x)b_{-4}c_0 + \xi^1(x)b_{-3}c_{-1} \\ & \left. + \xi^2(x)b_{-2}c_{-2} + \xi^3(x)b_{-1}c_{-3} + \xi^4(x)b_{-2}b_{-1}c_{-1}c_0 \right\} c_1 |k\rangle, \end{aligned} \quad (1.22)$$

$$\begin{aligned} \Lambda_4 = & \left\{ -i\epsilon_{\mu\nu\alpha}^0(x)\alpha_{-1}^\mu\alpha_{-1}^\nu\alpha_{-1}^\alpha b_{-1} - \epsilon_{\mu\nu}^1(x)\alpha_{-2}^\mu\alpha_{-1}^\nu b_{-1} - \epsilon_{\mu\nu}^2(x)\alpha_{-1}^\mu\alpha_{-1}^\nu b_{-2} + i\epsilon_\mu^3(x)\alpha_{-3}^\mu b_{-1} \right. \\ & + i\epsilon_\mu^4(x)\alpha_{-2}^\mu b_{-2} + i\epsilon_\mu^5(x)\alpha_{-1}^\mu b_{-3} + i\epsilon_\mu^6(x)\alpha_{-1}^\mu b_{-2}b_{-1}c_0 + \epsilon^4(x)b_{-4} \\ & \left. + \epsilon^5(x)b_{-3}b_{-1}c_0 + \epsilon^6(x)b_{-2}b_{-1}c_{-1} \right\} |\Omega\rangle, \end{aligned} \quad (1.23)$$

The transformations for the matter part are

$$\delta D_{\mu\nu\alpha\beta} = -\partial_{(\beta}\epsilon_{\mu\nu\alpha)}^0 - \frac{1}{2}\epsilon_{(\mu\nu}^2\eta_{\alpha\beta)}, \quad (1.24a)$$

$$\delta D_{\mu\nu\alpha} = -\partial_{(\mu}\epsilon_{|\alpha|\nu)}^1 - \partial_\alpha\epsilon_{\nu\mu}^2 + 3\epsilon_{\mu\nu\alpha}^0 - \frac{1}{2}\epsilon_\alpha^4\eta_{\nu\mu} - \epsilon_{(\mu}^5\eta_{\nu)\alpha}, \quad (1.24b)$$

$$\delta D_{[\mu\nu]}^1 = -\partial_{[\mu}\epsilon_{\nu]}^3 - \partial_{[\nu}\epsilon_{\mu]}^5 + 2\epsilon_{[\nu\mu]}^1, \quad (1.24c)$$

$$\delta D_{(\mu\nu)}^1 = -\partial_{(\mu}\epsilon_{\nu)}^3 - \partial_{(\nu}\epsilon_{\mu)}^5 + 2\epsilon_{(\nu\mu)}^1 + 2\epsilon_{\mu\nu}^2 - \epsilon^4\eta_{\mu\nu}, \quad (1.24d)$$

$$\delta D_{\mu\nu}^0 = -\partial_{(\mu}\epsilon_{\nu)}^4 + \epsilon_{(\nu\mu)}^1 - \frac{1}{2}\epsilon^4\eta_{\mu\nu}, \quad (1.24e)$$

$$\delta D_\mu = -\partial_\mu\epsilon^4 + 3\epsilon_\mu^3 + 2\epsilon_\mu^4 + \epsilon_\mu^5. \quad (1.24f)$$

It can be checked from the above equations that only $D_{\mu\nu\alpha\beta}$ and mixed-symmetric $D_{\mu\nu\alpha}$ cannot be gauged away, which is consistent with the result of the first quantized approach in the previous sections. That is, the spin-two and scalar positive-norm physical propagating modes can be gauged to $D_{\mu\nu\alpha\beta}$ and mixed symmetric $D_{\mu\nu\alpha}$. In fact, $D_{\mu\nu\alpha}$, $D_{[\mu\nu]}^1$, $D_{(\mu\nu)}^1$, $D_{\mu\nu}^0$ and D_μ can be gauged away by $\epsilon_{\mu\nu\alpha}^0$, $\epsilon_{[\mu\nu]}^1$, $\epsilon_{(\mu\nu)}^1$, $\epsilon_{\mu\nu}^2$ and one of the vector parameters, say ϵ_μ^3 . The rest, ϵ_μ^4 , ϵ_μ^5 and ϵ^4 are gauge artifacts of $D_{\mu\nu\alpha\beta}$ and mixed-symmetric $D_{\mu\nu\alpha}$.

The transformation for the ghost part are

$$\delta\xi_{\mu\nu\alpha} = \frac{1}{2}(\partial^2 - 6)\epsilon_{\mu\nu\alpha}^0 - \frac{1}{2}\epsilon_{(\mu}^6\eta_{\nu\alpha)}, \quad (1.25a)$$

$$\delta\xi_{[\mu\nu]}^0 = \frac{1}{2}(\partial^2 - 6)\epsilon_{[\mu\nu]}^1 - \partial_{[\mu}\epsilon_{\nu]}^6, \quad (1.25b)$$

$$\delta\xi_{(\mu\nu)}^0 = \frac{1}{2}(\partial^2 - 6)\epsilon_{(\mu\nu)}^1 - \partial_{(\mu}\epsilon_{\nu)}^6 + \epsilon^5\eta_{\mu\nu}, \quad (1.25c)$$

$$\delta\xi_{\mu\nu}^1 = \frac{1}{2}(\partial^2 - 6)\epsilon_{\mu\nu}^2 + \partial_{(\mu}\epsilon_{\nu)}^6, \quad (1.25d)$$

$$\delta\xi_{\mu\nu}^2 = -3\partial^\alpha\epsilon_{\mu\nu\alpha}^0 - 2\epsilon_{(\mu\nu)}^1 - 3\epsilon_{\mu\nu}^2 - \frac{1}{2}\epsilon^6\eta_{\mu\nu}, \quad (1.25e)$$

$$\delta\xi_\mu^0 = \frac{1}{2}(\partial^2 - 6)\epsilon_\mu^3 - \partial_\mu\epsilon^5 + \epsilon_\mu^6, \quad (1.25f)$$

$$\delta\xi_\mu^1 = \frac{1}{2}(\partial^2 - 6)\epsilon_\mu^4 - \epsilon_\mu^6, \quad (1.25g)$$

$$\delta\xi_\mu^2 = \frac{1}{2}(\partial^2 - 6)\epsilon_\mu^5 + \partial_\mu\epsilon^5 - \epsilon_\mu^6, \quad (1.25h)$$

$$\delta\xi_\mu^3 = -\partial^\nu\epsilon_{\mu\nu}^1 - \partial_\mu\epsilon^6 - 3\epsilon_\mu^3 - 3\epsilon_\mu^4, \quad (1.25i)$$

$$\delta\xi_\mu^4 = 2\partial^\nu\epsilon_{\mu\nu}^2 + \partial_\mu\epsilon^6 - 2\epsilon_\mu^4 - 4\epsilon_\mu^5 - 2\epsilon_\mu^6, \quad (1.25j)$$

$$\delta\xi_\mu^5 = -2\partial^\nu\epsilon_{\mu\nu}^1 - 3\epsilon_\mu^3 - 5\epsilon_\mu^5 + 4\epsilon_\mu^6 + 3\epsilon_{\mu\nu}^{0\nu}, \quad (1.25k)$$

$$\delta\xi^0 = \frac{1}{2}(\partial^2 - 6)\epsilon^4 - 2\epsilon^5, \quad (1.25l)$$

$$\delta\xi^1 = -\partial^\mu\epsilon_\mu^5 - 5\epsilon^4 - 2\epsilon^5 - \epsilon^6, \quad (1.25m)$$

$$\delta\xi^2 = -2\partial^\mu\epsilon_\mu^4 - 6\epsilon^4 - 3\epsilon^6 + \epsilon_\mu^2{}^\mu, \quad (1.25n)$$

$$\delta\xi^3 = -3\partial^\mu\epsilon_\mu^3 - 7\epsilon^4 + 6\epsilon^5 + 5\epsilon^6 + 2\epsilon_\mu^1{}^\mu, \quad (1.25o)$$

$$\delta\xi^4 = \frac{1}{2}(\partial^2 - 6)\epsilon^6 + \partial^\mu\epsilon_\mu^6 + 4\epsilon^5. \quad (1.25p)$$

There are nine on-mass-shell conditions, which contains a symmetric spin three, an anti-symmetric spin two, two symmetric spin two, three vector and two scalar fields, and seven gauge conditions which amounts to sixteen equations in Eq.(1.25a) to Eq.(1.25p). This is consistent with counting from ZNS listed in the table. Three traceless conditions read from ZNS corresponds to the three equations involving $\delta\xi_{\mu\nu}{}^\nu$, $\delta\xi_\mu^{0\mu}$, $\delta\xi_\mu^{1\mu}$ which are contained in Eq.(1.25a), Eq.(1.25c), and Eq.(1.25d).

It is important to note that the transformation for the matter parts, Eq.(1.21a) to Eq.(1.21d) and Eq.(1.24a) to Eq.(1.24f), are the same as the calculation [10] based on the chordal gauge transformation of free covariant string field theory constructed by Banks and

Peskin [104]. The Chordal gauge transformation can be written in the following form

$$\delta\Phi[X(\sigma)] = \sum_{n>0} L_{-n}\Phi_n[X(\sigma)] \quad (1.26)$$

where $\Phi[X(\sigma)]$ is the string field and $\Phi_n[X(\sigma)]$ are gauge parameters which are functions of $X[\sigma]$ only and free of ghost fields. This is because the pure ghost part of Q_B in Eq.(1.18) does not contribute to the transformation of matter background fields. It is interesting to note that the r.h.s. of Eq.(1.26) is in the form of off-shell spurious states [9] in the OCFQ approach. They become ZNS on imposing the physical and on-shell state condition.

Finally, it can be shown that the number of scalar ZNS at n -th massive level ($n \geq 3$) is at least the sum of those at $(n-2)$ -th and $(n-1)$ -th massive levels. So positive-norm scalar modes at n -th level, if they exist, will be decoupled according to our decoupling conjecture.

The decoupling of these scalars has important implication on Sen's conjectures on the decay of open string tachyon [105]. Since all scalars on D -brane including tachyon get non-zero vev. in the false vacuum, they will decay together with tachyon and disappear eventually to the true closed string vacuum. As the scalar states together with higher tensor states form a large gauge multiplet at each mass level, and its scattering amplitudes are fixed by the tensor fields, these tensor fields of open string ($D25$ -brane) will accompany the decay process. This means that the whole D -brane could disappear to the true closed string vacuum.

II. CALCULATION OF HIGHER MASSIVE ZNS

Since ZNS are the most important key to generate stringy symmetries, in this chapter we give a simplified method [94] to generate two types of ZNS in the old covariant first quantized (OCFQ) spectrum of open bosonic string. ZNS up to the fourth massive level and general formulas of some zero-norm tensor states at arbitrary mass levels will be calculated.

The vertex operator of a physical state of open bosonic string

$$|\Psi\rangle = \sum C_{\mu_1 \dots \mu_m} \alpha_{-n_1}^{\mu_1} \dots \alpha_{-n_m}^{\mu_m} |0; k\rangle, [\alpha_m^\mu, \alpha_n^\nu] = m\eta^{\mu\nu} \delta_{m+n} \quad (2.1)$$

is given by [100]

$$\Psi(z) = \sum C_{\mu_1 \dots \mu_m} N_m : \prod (\partial_z^{n_j} x^{\mu_j}) e^{ik \cdot X(z)} :, \quad (2.2)$$

where $N_m = i^m \prod \{(n_j - 1)!\}^{-1}$. In the OCFQ spectrum, physical states in Eq.(2.1) are subject to the following Virasoro conditions

$$(L_0 - 1) |\Psi\rangle = 0, L_1 |\Psi\rangle = L_2 |\Psi\rangle = 0, \quad (2.3)$$

where

$$L_m = \frac{1}{2} \sum_{-\infty}^{\infty} : \alpha_{m-n} \cdot \alpha_n : \quad (2.4)$$

and $\alpha_0 \equiv k$. The solutions of Eq.(2.3) include positive-norm propagating states and two types of ZNS in Eq.(1.1) and Eq.(1.2) which can be derived from Kac determinant in conformal field theory. While type I states have zero-norm at any spacetime dimension, type II states have zero-norm *only* at $D = 26$. The existence of type II ZNS signals the importance of ZNS in the structure of the theory of string. It is straightforward to solve positive-norm state solutions of Eq.(2.3) for some low-lying states, but soon becomes practically unmanageable. The authors of Ref [106] gave a simple prescription to solve the positive-norm state solutions of Eq.(2.3). The strategy is to apply the Virasoro conditions only to purely transverse states, so that the ZNS will be removed at the very beginning. This prescription simplified a lot of computation although some complexities remained for low spin states at higher levels. Our aim in this chapter is to generate ZNS in Eq.(1.1) and Eq.(1.2) so that all physical state solutions of Eq.(2.3) will be completed.

Let's first assume we are given positive-norm state solutions of some mass level n . The number of positive-norm degree of freedom at mass level n ($M^2 = 2(n - 1)$) is given by $N_{24}(n)$, where [107]

$$N_D(n) = \frac{1}{2\pi i} \oint \frac{dx}{x^{n+1}} (\prod_{k=1}^{\infty} \frac{1}{1 - x^k})^D. \quad (2.5)$$

On the other hand, the number of physical state degree of freedom is given by $N_{25}(n)$ in view of the constraints in Eq.(2.3). The discrepancy is of course due to physical ZNS given by solutions of Eq.(1.1) and Eq.(1.2). That is, among 25 chains of α_m^μ oscillators, one chain forms ZNS. Thus we can easily tabulate Young diagrams of ZNS at each mass level given Young diagrams of positive-norm states at the same mass level calculated by the simplified

prescription in [106]. For example, positive-norm state $\square\square\square\square$ at mass level $n = 4$ gives ZNS $\square\square\square + \square\square + \square + \bullet$, positive-norm state $\square\square$ gives ZNS $\square + \square\square + \square$ and positive-norm state \square gives ZNS $\square + \bullet$. This completes the ZNS at mass level $n = 4$. Young diagrams of ZNS up to mass level $M^2 = 10$, together with positive-norm states calculated in [106], will be listed in the later part of this chapter. A consistent check of counting of ZNS by using background ghost fields in WSFT was given in [14].

To explicitly calculate ZNS is another complicated issue. Suppose we are given some low-lying positive-norm state solutions. It is interesting to see the similarity between Eq.(2.3) and Eq.(1.1) and Eq.(1.2) for $|x\rangle$ and $|\tilde{x}\rangle$. The only difference is the "mass shift" of L_0 equations. As is well-known, the L_1 and L_2 equations give the transverse and traceless conditions on the spin polarization. It turns out that, in many cases, the L_1 and L_2 equations will not refer to the L_0 equation or on-mass-shell condition. In these cases, a positive-norm state solution for $|\Psi\rangle$ at mass level n will give a ZNS solution $L_{-1}|x\rangle$ at mass level $n + 1$ simply by taking $|x\rangle = |\Psi\rangle$ and shifting k^2 by one unit. Similarly, one can easily get a type II ZNS $(L_{-2} + \frac{3}{2}L_{-1}^2)|\tilde{x}\rangle$ at mass level $n + 2$ simply by taking $|\tilde{x}\rangle = |\Psi\rangle$ and shifting k^2 by two units. For those cases where L_1 and L_2 equations do refer to L_0 equation, our prescription needs to be modified. We will give some examples to illustrate this method. Note that once we generate a ZNS, it soon becomes a candidate of physical state $|\Psi\rangle$ to generate two new ZNS at even higher levels.

1. The first ZNS begin at $k^2 = 0$. This state is suggested from the positive-norm tachyon state $|0, k\rangle$ with $k^2 = 2$. Taking $|x\rangle = |0, k\rangle$ and shifting k^2 by one unit to $k^2 = 0$, we get a type I ZNS.

$$L_{-1}|x\rangle = k \cdot \alpha_{-1}|0, k\rangle; |x\rangle = |0, k\rangle, -k^2 = M^2 = 0. \quad (2.6)$$

2. At the first massive level $k^2 = -2$, tachyon suggests a type II ZNS

$$(L_{-2} + \frac{3}{2}L_{-1}^2)|\tilde{x}\rangle = [\frac{1}{2}\alpha_{-1} \cdot \alpha_{-1} + \frac{5}{2}k \cdot \alpha_{-2} + \frac{3}{2}(k \cdot \alpha_{-1})^2]|0, k\rangle; |\tilde{x}\rangle = |0, k\rangle, -k^2 = 2. \quad (2.7)$$

Positive-norm massless vector state suggests a type I ZNS

$$L_{-1}|x\rangle = [\theta \cdot \alpha_{-2} + (k \cdot \alpha_{-1})(\theta \cdot \alpha_{-1})]|0, k\rangle; |x\rangle = \theta \cdot \alpha_{-1}|0, k\rangle, -k^2 = 2, \theta \cdot k = 0. \quad (2.8)$$

However, massless singlet ZNS Eq.(2.6) does not give a type I ZNS at the first massive level $k^2 = -2$ since L_1 equation on state Eq.(2.6) refers to L_0 equation, $k^2 = 0$. This means that L_1 will not annihilate state Eq.(2.6) if one shifts the mass to $k^2 = -2$.

3. At the second massive level $k^2 = -4$, positive-norm massless vector state suggests a type II ZNS

$$\begin{aligned} (L_{-2} + \frac{3}{2}L_{-1}^2) |\tilde{x}\rangle &= \{4\theta \cdot \alpha_{-3} + \frac{1}{2}(\alpha_{-1} \cdot \alpha_{-1})(\theta \cdot \alpha_{-1}) + \frac{5}{2}(k \cdot \alpha_{-2})(\theta \cdot \alpha_{-1}) \\ &\quad + \frac{3}{2}(k \cdot \alpha_{-1})^2(\theta \cdot \alpha_{-1}) + 3(k \cdot \alpha_{-1})(\theta \cdot \alpha_{-2})\} |0, k\rangle; \\ |\tilde{x}\rangle &= \theta \cdot \alpha_{-1} |0, k\rangle, -k^2 = 4, k \cdot \theta = 0. \end{aligned} \quad (2.9)$$

However, massless singlet ZNS Eq.(2.6) does not give a type I ZNS at mass level $k^2 = -4$ for the same reason stated after Eq.(2.8). Positive-norm spin-two state at $k^2 = -2$ suggests a type I ZNS

$$\begin{aligned} L_{-1} |x\rangle &= [2\theta_{\mu\nu}\alpha_{-1}^\mu\alpha_{-2}^\nu + k_\lambda\theta_{\mu\nu}\alpha_{-1}^{\lambda\mu\nu}] |0, k\rangle; |x\rangle = \theta_{\mu\nu}\alpha_{-1}^{\mu\nu} |0, k\rangle, -k^2 = 4, \\ k \cdot \theta &= \eta^{\mu\nu}\theta_{\mu\nu} = 0, \theta_{\mu\nu} = \theta_{\nu\mu}, \end{aligned} \quad (2.10)$$

where $\alpha_{-1}^{\lambda\mu\nu} \equiv \alpha_{-1}^\lambda\alpha_{-1}^\mu\alpha_{-1}^\nu$. Similar notations will be used in the rest of this paper. Vector ZNS with $k^2 = -2$ in Eq.(2.8) does not give a type I ZNS for the same reason stated after Eq.(2.8). In this case, however, one can modify $|x\rangle$ to be

$$\text{Ansatz: } |x\rangle = [a\theta \cdot \alpha_{-2} + b(k \cdot \alpha_{-1})(\theta \cdot \alpha_{-1})] |0, k\rangle; -k^2 = 4, \theta \cdot k = 0, \quad (2.11)$$

where a, b are undetermined constants. L_0 equation is then trivially satisfied and L_1, L_2 equations give $a : b = 2 : 1$. This gives a type I ZNS

$$\begin{aligned} L_{-1} |x\rangle &= [\frac{1}{2}(k \cdot \alpha_{-1})^2(\theta \cdot \alpha_{-1}) + 2\theta \cdot \alpha_{-3} + \frac{3}{2}(k \cdot \alpha_{-1})(\theta \cdot \alpha_{-2}) \\ &\quad + \frac{1}{2}(k \cdot \alpha_{-2})(\theta \cdot \alpha_{-1})] |0, k\rangle; -k^2 = 4, \theta \cdot k = 0. \end{aligned} \quad (2.12)$$

Similarly, we modify the singlet ZNS with $k^2 = -2$ in Eq.(9) to be

$$\text{Ansatz: } |x\rangle = [\frac{5}{2}ak \cdot \alpha_{-2} + \frac{1}{2}b\alpha_{-1} \cdot \alpha_{-1} + \frac{3}{2}c(k \cdot \alpha_{-1})^2] |0, k\rangle; -k^2 = 4, \quad (2.13)$$

where a , b and c are undetermined constants. L_1 and L_2 equations give

$$5a + b + 3k^2c = 0, 5k^2a + 13b + \frac{3}{2}k^2c = 0. \quad (2.14)$$

For $k^2 = -4$, we have $a : b : c = 5 : 9 : \frac{17}{6}$. This gives a type I ZNS

$$\begin{aligned} L_{-1} |x\rangle &= [\frac{17}{4}(k \cdot \alpha_{-1})^3 + \frac{9}{2}(k \cdot \alpha_{-1})(\alpha_{-1} \cdot \alpha_{-1}) + 9(\alpha_{-1} \cdot \alpha_{-2}) \\ &\quad + 21(k \cdot \alpha_{-1})(k \cdot \alpha_{-2}) + 25(k \cdot \alpha_{-3})] |0, k\rangle; \\ -k^2 &= 4. \end{aligned} \quad (2.15)$$

This completes the four ZNS at the second massive level. It is interesting to note that the Young tableau of ZNS at level $M^2 = 4$ are the sum of those of all physical states at two lower levels, $M^2 = 2$ and $M^2 = 0$, *except* the singlet ZNS due to the dependence of L_1 and L_2 equations on L_0 condition in state Eq.(2.6). For those cases that L_1 and L_2 equations not referring to L_0 condition, our construction gives us a very simple way to calculate ZNS at any mass level n given those of positive-norm states at lower levels constructed by the simplified method in Ref [106]. When the modified method was needed to calculate a higher mass level ZNS from a lower mass level physical state like Eq.(2.6), an inconsistency may result and one gets no ZNS. This explains the discrepancy of singlet ZNS at levels $M^2 = 2, 4, 8$ and a vector ZNS at level $M^2 = 10$.

4. Similar method can be used to calculate ZNS at level $M^2 = 6$. We will just list some examples here. They are (from now on, unless otherwise stated, each spin polarization is assumed to be transverse, traceless and is symmetric with respect to each group of indices as in Ref [106])

$$L_{-1} |x\rangle = \theta_{\mu\nu\lambda}(k_\beta \alpha_{-1}^{\mu\nu\lambda\beta} + 3\alpha_{-1}^{\mu\nu} \alpha_{-2}^\lambda) |0, k\rangle; |x\rangle = \theta_{\mu\nu\lambda} \alpha_{-1}^{\mu\nu\lambda} |0, k\rangle, \quad (2.16)$$

$$L_{-1} |x\rangle = [k_\lambda \theta_{\mu\nu} \alpha_{-1}^{\mu\lambda} \alpha_{-2}^\nu + 2\theta_{\mu\nu} \alpha_{-1}^\mu \alpha_{-3}^\nu] |0, k\rangle; |x\rangle = \theta_{\mu\nu} \alpha_{-1}^\mu \alpha_{-2}^\nu |0, k\rangle, \text{ where } \theta_{\mu\nu} = -\theta_{\nu\mu}, \quad (2.17)$$

$$\begin{aligned}
L_{-1} |x\rangle &= [2\theta_{\mu\nu}\alpha_{-2}^{\mu\nu} + 4\theta_{\mu\nu}\alpha_{-1}^{\mu}\alpha_{-3}^{\nu} + 2(k_{\lambda}\theta_{\mu\nu} + k_{(\lambda}\theta_{\mu\nu)})\alpha_{-1}^{\lambda\mu}\alpha_{-2}^{\nu} + \frac{2}{3}k_{\lambda}k_{\beta}\theta_{\mu\nu}\alpha_{-1}^{\mu\nu\lambda\beta}] |0, k\rangle; \\
|x\rangle &= [2\theta_{\mu\nu}\alpha_{-1}^{\mu}\alpha_{-2}^{\nu} + \frac{2}{3}k_{\lambda}\theta_{\mu\nu}\alpha_{-1}^{\mu\nu\lambda}] |0, k\rangle
\end{aligned} \tag{2.18}$$

and

$$\begin{aligned}
(L_{-2} + \frac{3}{2}L_{-1}^2) |\tilde{x}\rangle &= [3\theta_{\mu\nu}\alpha_{-2}^{\mu\nu} + 8\theta_{\mu\nu}\alpha_{-1}^{\mu}\alpha_{-3}^{\nu} + (k_{\lambda}\theta_{\mu\nu} + \frac{15}{2}k_{(\lambda}\theta_{\mu\nu)})\alpha_{-1}^{\lambda\mu}\alpha_{-2}^{\nu} \\
&\quad + (\frac{1}{2}\eta_{\lambda\beta}\theta_{\mu\nu} + \frac{3}{2}k_{\lambda}k_{\beta}\theta_{\mu\nu})\alpha_{-1}^{\mu\nu\lambda\beta}] |0, k\rangle; \\
|\tilde{x}\rangle &= \theta_{\mu\nu}\alpha_{-1}^{\mu\nu} |0, k\rangle.
\end{aligned} \tag{2.19}$$

Note that $|x\rangle$ in Eq.(2.18) has been modified as we did for Eq.(2.11). To further illustrate our method, we calculate the type I singlet ZNS from Eq.(2.15) as following

$$\begin{aligned}
\text{Ansatz : } |x\rangle &= [a(k \cdot \alpha_{-1})^3 + b(k \cdot \alpha_{-1})(\alpha_{-1} \cdot \alpha_{-1}) + c(k \cdot \alpha_{-1})(k \cdot \alpha_{-2}) \\
&\quad + d(\alpha_{-1} \cdot \alpha_{-2}) + f(k \cdot \alpha_{-3})] |0, k\rangle; \\
-k^2 &= 6.
\end{aligned} \tag{2.20}$$

The L_1 and L_2 equations can be easily used to determine $a : b : c : d : f = 37 : 72 : 261 : 216 : 450$. This gives the type I singlet ZNS

$$\begin{aligned}
L_{-1} |x\rangle &= [a(k \cdot \alpha_{-1})^4 + b(k \cdot \alpha_{-1})^2(\alpha_{-1} \cdot \alpha_{-1}) + (2b + d)(k \cdot \alpha_{-1})(\alpha_{-1} \cdot \alpha_{-2}) \\
&\quad + (c + 3a)(k \cdot \alpha_{-1})^2(k \cdot \alpha_{-2}) + c(k \cdot \alpha_{-2})^2 + d(\alpha_{-2} \cdot \alpha_{-2}) + b(k \cdot \alpha_{-2})(\alpha_{-1} \cdot \alpha_{-1}) \\
&\quad + (2c + f)(k \cdot \alpha_{-3})(k \cdot \alpha_{-1}) + 2d(\alpha_{-1} \cdot \alpha_{-3}) + 3f(k \cdot \alpha_{-4})] |0, k\rangle, \\
-k^2 &= 6.
\end{aligned} \tag{2.21}$$

5. We list relevant ZNS at level $M^2 = 8$ from the known positive-norm states and ZNS at level $M^2 = 4, 6$. They are

$$L_{-1} |x\rangle = (k_{\beta}\theta_{\mu\nu\lambda\gamma}\alpha_{-1}^{\mu\nu\lambda\gamma\beta} + 4\theta_{\mu\nu\lambda\gamma}\alpha_{-1}^{\mu\nu\lambda}\alpha_{-2}^{\gamma}) |0, k\rangle; |x\rangle = \theta_{\mu\nu\lambda\gamma}\alpha_{-1}^{\mu\nu\lambda\gamma} |0, k\rangle, \tag{2.22}$$

$$\begin{aligned}
L_{-1} |x\rangle &= \theta_{\mu\nu\lambda} \left[\frac{3}{4} k_\beta k_\gamma \alpha_{-1}^{\mu\nu\lambda\gamma\beta} + 3k_\beta \alpha_{-1}^{\mu\nu\beta} \alpha_{-2}^\lambda + 3k_\beta \alpha_{-1}^{(\mu\nu\lambda} \alpha_{-2}^{\beta)} + 6\alpha_{-1}^{(\mu} \alpha_{-2}^{\nu\lambda)} \right. \\
&\quad \left. + 6\alpha_{-1}^{(\mu\nu} \alpha_{-3}^{\lambda)} \right] |0, k\rangle; \quad |x\rangle = \theta_{\mu\nu\lambda} \left(\frac{3}{4} k_\beta \alpha_{-1}^{\mu\nu\lambda\beta} + 3\alpha_{-1}^{\mu\nu} \alpha_{-2}^\lambda \right) |0, k\rangle,
\end{aligned} \tag{2.23}$$

$$\begin{aligned}
(L_{-2} + \frac{3}{2} L_{-1}^2) |\tilde{x}\rangle &= \theta_{\mu\nu\lambda} \left[\left(\frac{3}{2} k_\beta k_\gamma + \frac{1}{2} \eta_{\gamma\beta} \right) \alpha_{-1}^{\mu\nu\lambda\beta\gamma} + k_\gamma \left(\frac{1}{2} \alpha_{-1}^{\mu\nu\lambda} \alpha_{-2}^\gamma + 8\alpha_{-1}^{(\mu\nu\lambda} \alpha_{-2}^{\gamma)} \right) \right. \\
&\quad \left. + 3\alpha_{-1}^{(\mu} \alpha_{-2}^{\nu\lambda)} + 6\alpha_{-1}^{(\mu\nu} \alpha_{-3}^{\lambda)} \right] |0, k\rangle; \\
|\tilde{x}\rangle &= \theta_{\mu\nu\lambda} \alpha_{-1}^{\mu\nu\lambda} |0, k\rangle,
\end{aligned} \tag{2.24}$$

$$\begin{aligned}
L_{-1} |x\rangle &= \theta_{\mu\nu,\lambda} (k_\gamma \alpha_{-1}^{\gamma\mu\nu} \alpha_{-2}^\lambda + 2\alpha_{-1}^\mu \alpha_{-2}^{\nu\lambda} + 2\alpha_{-1}^{\mu\nu} \alpha_{-3}^\lambda) |0, k\rangle; \\
|x\rangle &= \theta_{\mu\nu,\lambda} \alpha_{-1}^{\mu\nu} \alpha_{-2}^\lambda |0, k\rangle, \text{ where } \theta_{\mu\nu,\lambda} \text{ is mixed symmetric,}
\end{aligned} \tag{2.25}$$

$$\begin{aligned}
L_{-1} |x\rangle &= \theta_{\mu\nu} \left(\frac{3}{4} k_\beta k_\lambda \alpha_{-1}^{\beta\lambda\mu} \alpha_{-2}^\nu + 4k_\lambda \alpha_{-1}^{\lambda\mu} \alpha_{-3}^\nu + \frac{3}{4} k_\lambda \alpha_{-1}^\mu \alpha_{-2}^{\nu\lambda} + 2\alpha_{-2}^\mu \alpha_{-3}^\nu + 6\alpha_{-1}^\mu \alpha_{-4}^\nu \right) |0, k\rangle; \\
|x\rangle &= \left(\frac{3}{4} k_\lambda \alpha_{-1}^{\lambda\mu} \alpha_{-2}^\nu + 2\alpha_{-1}^\mu \alpha_{-3}^\nu \right) |0, k\rangle, \text{ where } \theta_{\mu\nu} = -\theta_{\nu\mu},
\end{aligned} \tag{2.26}$$

and

$$\begin{aligned}
(L_{-2} + \frac{3}{2} L_{-1}^2) |\tilde{x}\rangle &= \theta_{\mu\nu} \left[\left(\frac{3}{2} k_\gamma k_\lambda + \frac{1}{2} \eta_{\gamma\lambda} \right) \alpha_{-1}^{\gamma\lambda\mu} \alpha_{-2}^\nu + 6k_\lambda \alpha_{-1}^{\lambda\mu} \alpha_{-3}^\nu + \frac{5}{2} k_\lambda \alpha_{-1}^\mu \alpha_{-2}^{\nu\lambda} \right. \\
&\quad \left. + 2\alpha_{-2}^\mu \alpha_{-3}^\nu + \alpha_{-1}^\mu \alpha_{-4}^\nu \right] |0, k\rangle, \quad |\tilde{x}\rangle = \theta_{\mu\nu} \alpha_{-1}^\mu \alpha_{-2}^\nu |0, k\rangle, \text{ where } \theta_{\mu\nu} = -\theta_{\nu\mu}.
\end{aligned} \tag{2.27}$$

Note that the modified method was used in Eq.(2.23) and Eq.(2.26).

6. Finally, we calculate general formulas of some zero-norm tensor states at arbitrary mass levels by making use of general formulas of some positive-norm states listed in Ref [106].

a.

$$L_{-1} \theta_{\mu_1 \dots \mu_m} \alpha_{-1}^{\mu_1 \dots \mu_m} |0, k\rangle = \theta_{\mu_1 \dots \mu_m} (k_\lambda \alpha_{-1}^{\lambda \mu_1 \dots \mu_m} + m \alpha_{-2}^{\mu_1} \alpha_{-1}^{\mu_2 \dots \mu_m}) |0, k\rangle, \tag{2.28}$$

where $-k^2 = M^2 = 2m, m = 0, 1, 2, 3, \dots$. For example, $m = 0, 1$ give Eq.(2.6) and Eq.(2.8).

b.

$$\begin{aligned}
& (L_{-2} + \frac{3}{2}L_{-1}^2)\theta_{\mu_1 \dots \mu_m} \alpha_{-1}^{\mu_1 \dots \mu_m} |0, k\rangle \\
& = \{\theta_{\mu_1 \dots \mu_m} [(\frac{3}{2}k_\nu k_\lambda + \frac{1}{2}\eta_{\nu\lambda})\alpha_{-1}^{\nu\lambda\mu_1 \dots \mu_m} + \frac{3}{2}m(m-1)\alpha_{-2}^{\mu_1\mu_2} \alpha_{-1}^{\mu_3 \dots \mu_m} \\
& + (1+3m)\alpha_{-1}^{\mu_1 \dots \mu_{m-1}} \alpha_{-3}^{\mu_m}] + [\frac{3}{2}(m+1)k_{(\lambda}\theta_{\mu_1 \dots \mu_m)} + \frac{3}{2}mk_{\mu_m}\theta_{\mu_1 \dots \mu_{m-1}\lambda}]\} \\
& \alpha_{-1}^{\mu_1 \dots \mu_m} \alpha_{-2}^\lambda \} |0, k\rangle, \tag{2.29}
\end{aligned}$$

where $-k^2 = M^2 = 2m+2, m=0, 1, 2, \dots$. For example, $m=0, 1$ give Eq.(2.7) and Eq.(2.9).

c.

$$\begin{aligned}
& L_{-1}\theta_{\mu_1 \dots \mu_{m-2}, \mu_{m-1}} \alpha_{-1}^{\mu_1 \dots \mu_{m-2}} \alpha_{-2}^{\mu_{m-1}} |0, k\rangle \\
& = \theta_{\mu_1 \dots \mu_{m-2}, \mu_{m-1}} [k_\lambda \alpha_{-1}^{\lambda\mu_1 \dots \mu_{m-2}} \alpha_{-2}^{\mu_{m-1}} + (m-2)\alpha_{-1}^{\mu_1\mu_{m-3}} \alpha_{-2}^{\mu_{m-2}\mu_m} \\
& + 2\alpha_{-1}^{\mu_1 \dots \mu_{m-2}} \alpha_{-2}^{\mu_{m-1}}] |0, k\rangle, \quad \boxed{\begin{array}{c} \dots \end{array}} \tag{2.30}
\end{aligned}$$

where $-k^2 = M^2 = 2m, m=3, 4, 5, \dots$. For example, $m=3, 4$ give Eq.(2.17) and Eq.(2.25).

d.

$$\begin{aligned}
& (L_{-2} + \frac{3}{2}L_{-1}^2)\theta_{\mu_1 \dots \mu_{m-2}, \mu_{m-1}} \alpha_{-1}^{\mu_1 \dots \mu_{m-2}} \alpha_{-2}^{\mu_{m-1}} |0, k\rangle \\
& = \theta_{\mu_1 \dots \mu_{m-2}, \mu_{m-1}} [(\frac{3}{2}k_\lambda k_\nu + \frac{1}{2}\eta_{\lambda\nu})\alpha_{-1}^{\mu_1 \dots \mu_{m-2}\lambda\nu} \alpha_{-2}^{\mu_{m-1}} + 6k_\lambda \alpha_{-1}^{\mu_1 \dots \mu_{m-2}\lambda} \alpha_{-3}^{\mu_{m-1}} \\
& + (\frac{3}{2}m-2)k_\lambda \alpha_{-1}^{\mu_1 \dots \mu_{m-2}} \alpha_{-2}^{\mu_{m-1}\lambda} + 2(m-2)\alpha_{-1}^{\mu_1 \dots \mu_{m-3}} \alpha_{-2}^{\mu_{m-2}} \alpha_{-3}^{\mu_{m-1}} + 11\alpha_{-1}^{\mu_1 \dots \mu_{m-2}} \alpha_{-4}^{\mu_{m-1}} \\
& + k_\lambda \alpha_{-1}^{\mu_1 \dots \mu_{m-3}\lambda} \alpha_{-2}^{\mu_{m-2}\mu_{m-1}} + (m-3)\alpha_{-1}^{\mu_1 \dots \mu_{m-4}} \alpha_{-2}^{\mu_{m-3}\mu_{m-2}\mu_{m-1}}] |0, k\rangle, \quad \boxed{\begin{array}{c} \dots \end{array}} \tag{2.31}
\end{aligned}$$

where $-k^2 = M^2 = 2m+2, m=3, 4, 5, \dots$. For example, $m=3$ gives Eq.(2.27).

e.

$$\begin{aligned}
& L_{-1}\theta_{\mu_1 \dots \mu_{m-4}, \mu_{m-3}\mu_{m-2}} (\alpha_{-1}^{\mu_1 \dots \mu_{m-4}} \alpha_{-2}^{\mu_{m-3}\mu_{m-2}} - \frac{4}{3}\alpha_{-1}^{\mu_1 \dots \mu_{m-3}} \alpha_{-3}^{\mu_{m-2}}) \\
& = \theta_{\mu_1 \dots \mu_{m-4}, \mu_{m-3}\mu_{m-2}} [k_\lambda \alpha_{-1}^{\lambda\mu_1 \dots \mu_{m-4}} \alpha_{-2}^{\mu_{m-3}\mu_{m-2}} + (m-4)\alpha_{-1}^{\mu_1 \dots \mu_{m-3}} \alpha_{-2}^{\mu_{m-4}\mu_{m-3}\mu_{m-2}} \\
& + \frac{16}{3}\alpha_{-1}^{\mu_1 \dots \mu_{m-4}} \alpha_{-3}^{\mu_{m-3}} \alpha_{-2}^{\mu_{m-2}} + \frac{4}{3}k_\lambda \alpha_{-1}^{\lambda\mu_1 \dots \mu_{m-3}} \alpha_{-3}^{\mu_{m-2}} + 4\alpha_{-1}^{\mu_1 \dots \mu_{m-3}} \alpha_{-4}^{\mu_{m-4}}], \quad \boxed{\begin{array}{c} \dots \end{array}} \tag{2.32}
\end{aligned}$$

where $-k^2 = M^2 = 2m, m=5, 6, \dots$

f. The ZNS of Eq.(2.28) can be used to generate new type I ZNS by the modified method as following

$$\begin{aligned}
& L_{-1}\theta_{\mu_1\ldots\mu_m}\left(\frac{m}{m+1}k_\lambda\alpha_{-1}^{\lambda\mu_1\ldots\mu_m} + \alpha_{-2}^{\mu_1}\alpha_{-1}^{\mu_2\ldots\mu_m}\right)|0,k\rangle \\
& = \left[\frac{m}{m+1}k_\nu k_\lambda\theta_{\mu_1\ldots\mu_m}\alpha_{-1}^{\nu\lambda\mu_1\ldots\mu_m} + m(k_{(\lambda}\theta_{\mu_1\ldots\mu_m)} + k_\lambda\theta_{\mu_1\ldots\mu_m})\alpha_{-2}^{\mu_1}\alpha_{-1}^{\lambda\mu_2\ldots\mu_m}\right. \\
& \quad \left.+ m(m-1)\theta_{\mu_1\ldots\mu_m}\alpha_{-2}^{\mu_1\mu_2}\alpha_{-1}^{\mu_3\ldots\mu_m} + 2m\theta_{\mu_1\ldots\mu_m}\alpha_{-3}^{\mu_1}\alpha_{-1}^{\mu_2\ldots\mu_m}\right]|0,k\rangle, \tag{2.33}
\end{aligned}$$

where $-k^2 = M^2 = 2m + 2, m = 1, 2, 3, \dots$. For example, $m = 1, 2$ and 3 give Eq.(3.14), Eq.(2.18) and Eq.(2.23). Note that the coefficient of the first term in Eq.(2.33) has been modified to $\frac{m}{m+1}$. Similarly, new type II ZNS can also be constructed.

The Young tabulations of all physical states solutions of Eq.(2.3) up to level six, including two types of ZNS solutions of Eq.(1.1) and Eq.(1.2), are listed in the following table

massive level	positive-norm states	ZNS
$M^2 = -2$	•	
$M^2 = 0$	□	• (singlet)
$M^2 = 2$	□□	□, •
$M^2 = 4$	□□□, □	□□, $2 \times \square$, •
$M^2 = 6$	□□□□, □□, □□, •	□□□, □, $2 \times \square$, $3 \times \square$, $2 \times \bullet$
$M^2 = 8$	□□□□□, □□□, □□, □□□, □□, □	□□□□, □□, $2 \times \square$, $2 \times \square$, $4 \times \square$, $5 \times \square$, $3 \times \bullet$
$M^2 = 10$	□□□□□□, □□□□, □□□□, □□□□ □□, □□□, □□, □, $2 \times \square$, □, •	□□□□□, $2 \times \square$, □□□, $3 \times \square$, $4 \times \square$, $4 \times \square$, $7 \times \square$, $8 \times \square$, $6 \times \bullet$

Note that the Young tabulations of ZNS at level n are subset of the sum of all physical states at levels $n - 1$ and $n - 2$.

III. DISCRETE ZNS AND w_∞ SYMMETRY OF 2D STRING THEORY

For the $26D$ ($10D$) string theory, it is difficult to do calculations for higher mass string states and extract their symmetry structures which are valid for all energies. This is of course due to the high dimensionality of spacetime. One way to overcome this difficulty has been to probe high energy regimes of the theory and simplify the calculations. This will be done in part II and part III of this review. Another strategy was to study the toy string model, namely, $2D$ string theory or $c = 1$ $2D$ quantum gravity. The $2D$ string theory has been an important laboratory to study non-perturbative information of string theory. In the continuum Liouville approach [108], in addition to the massless tachyon mode, an infinite number of massive discrete momentum physical degrees of freedom were discovered

[109–113] and the target spacetime w_∞ symmetry and Ward identities were then identified [17–21].

In this chapter, we will derive the w_∞ symmetry structure from the ZNS point of view in the old covariant quantization scheme [22, 23]. This is in parallel with the works of [17] and [20, 21] where the ground ring structure of ghost number zero operators were identified in the BRST quantization. Moreover, the results we obtained will justify the idea of ZNS used in the $26D$ (or $10D$) theories as discussed previously.

Unlike the discrete Polyakov states, we will find that there are still an infinite number of continuum momentum ZNS in the massive levels of the $2D$ spectrum and it is very difficult to give a general formula for them just as in the case of $26D$ theory [11–13]. However, as far as the dynamics of the theory is concerned, only those ZNS with Polyakov discrete momenta are relevant. This is because all other ZNS are trivially decoupled from the correlation functions due to kinematic reason. Hence, we will only identify all discrete ZNS or discrete gauge states (DGS) in the spectrum. The higher the momentum is, the more numerous the DGS are found. In particular, we will give an explicit formula for one such set of DGS in terms of Schur polynomials. Finally, we can show that these DGS carry the w_∞ charges and serve as the symmetry parameters of the theory.

A. $2D$ string theory

1. ZNS in $2D$ string theory

We consider the two dimensional critical string action [108]

$$S = \frac{1}{8\pi} \int d^2\sigma \sqrt{\hat{g}} [g^{\mu\nu} (\partial_\mu X \partial_\nu X + \partial_\mu \phi \partial_\nu \phi) - Q \hat{R} \phi] \quad (3.1)$$

with ϕ being the Liouville field. For $c = 1$ theory Q , which represents the background charge of the Liouville field, is set to be $2\sqrt{2}$ so that the total anomalies cancels that from ghost contribution.

For simplicity here we consider only one of the chiral sectors, while the other sector (denoted by \bar{z}) is the same. The stress energy tensor is

$$T_{zz} = -\frac{1}{2}(\partial_z X)^2 - \frac{1}{2}(\partial_z \phi)^2 - \frac{1}{2}Q\partial_z^2 \phi. \quad (3.2)$$

If we define the mode expansion of $X^\mu = (\phi, X)$ by

$$\partial_z X^\mu = - \sum_{n=-\infty}^{\infty} z^{-n-1} (\alpha_n^0, i\alpha_n^1) \quad (3.3)$$

with the Minkowski metric $\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, $Q^\mu = (2\sqrt{2}, 0)$ and the zero mode $\alpha_0^\mu = f^\mu = (\epsilon, p)$, we find the Virasoro generators

$$\begin{aligned} L_n &= \left(\frac{n+1}{2} Q^\mu + f^\mu \right) \alpha_{\mu,n} + \frac{1}{2} \sum_{k \neq 0} : \alpha_{\mu,-k} \alpha_{n+k}^\mu : \quad n \neq 0, \\ L_0 &= \frac{1}{2} (Q^\mu + f^\mu) f_\mu + \sum_{k=1}^{\infty} : \alpha_{\mu,-k} \alpha_k^\mu : . \end{aligned} \quad (3.4)$$

The vacuum $|0\rangle$ is annihilated by all α_n^μ with $n > 0$. In the old covariant quantization, physical states $|\psi\rangle$ are those satisfy the condition

$$\begin{aligned} L_n |\psi\rangle &= 0 \quad \text{for} \quad n > 0, \\ L_0 |\psi\rangle &= |\psi\rangle . \end{aligned} \quad (3.5)$$

One can easily check that the two branches of massless tachyon

$$T^\pm(p) = e^{ipX + (\pm|p| - \sqrt{2})\phi} \quad (3.6)$$

are positive norm physical states. In the material gauge[6], it was also known that there exist discrete states [3] [10] ($J = \{0, 12, 1\dots\}$ and $M = \{-J, -J+1, \dots J\}$)

$$\psi_{J,M}^{(\pm)} \sim (H_-)^{J-M} \psi_{J,J}^{(\pm)} \sim (H_+)^{J+M} \psi_{J,-J}^{(\pm)}, \quad (3.7)$$

which are also positive norm physical states. In Eq.(3.7) $H_\pm = \int \frac{dz}{2\pi i} T^\pm(\pm\sqrt{2})$ are the zero modes of the ladder operators of the $SU(2)$ Kac-Moody currents at the self-dual radius in $c = 1$ $2d$ conformal field theory and $\psi_{J,\pm J}^{(\pm)} = T^{(\pm)}(\pm\sqrt{2}J)$. These exhaust all positive norm physical states. In this chapter we are interested in the discrete ZNS or discrete gauge states (DGS), i.e., the zero norm physical states at the same discrete momenta as those states in Eq.(3.7). We thus no longer restrict ourselves in the material gauge, and the Liouville field ϕ will play an important role in the following discussions.

In general, there are two types of ZNS in $2D$ string theory,

Type I:

$$|\psi\rangle = L_{-1} |\chi\rangle \quad \text{where} \quad L_m |\chi\rangle = 0 \quad m \geq 0, \quad (3.8)$$

Type II:

$$|\psi\rangle = \left(L_{-2} + \frac{3}{2} L_{-1}^2 \right) |\tilde{\chi}\rangle \quad \text{where} \quad L_m |\tilde{\chi}\rangle = 0 \quad m > 0, \quad (3.9)$$

$$(L_0 + 1) |\tilde{\chi}\rangle = 0.$$

They satisfy the physical state conditions Eq.(3.5), and have zero norm. It is important to note that state in Eq.(3.9) is a ZNS only when $Q = \sqrt{\frac{25-c}{3}}$, while the states in Eq.(3.8) are insensitive to this condition. In this section we will explicitly calculate the DGS at the two lowest mass levels. At mass level one (i.e. spin one), $f_\mu(f^\mu + Q^\mu) = 0$, only DGS of type I are found: $f_\mu \alpha_{-1}^\mu |f\rangle$, where $|f\rangle =: e^{ipX + \epsilon\phi} : |0\rangle$. The DGS $G_{1,0}^- =: \partial\phi e^{-2\sqrt{2}\phi} : |0\rangle$ corresponds to the momentum of $\psi_{1,0}^-$. There is no corresponding DGS for $\psi_{1,0}^+$.

At mass level two, $f_\mu(f^\mu + Q^\mu) = -2$, if $e_\mu(f^\mu + Q^\mu) = 0$ then the type I ZNS is

$$|\psi\rangle = \left[\frac{1}{2} (f_\mu e_\nu + e_\mu f_\nu) \alpha_{-1}^\mu \alpha_{-1}^\nu + e_\mu \alpha_{-2}^\mu \right] |f\rangle, \quad (3.10)$$

while the type II ZNS is

$$|\psi\rangle = \frac{1}{2} \left[(3f_\mu f_\nu + \eta_{\mu\nu}) \alpha_{-1}^\mu \alpha_{-1}^\nu + (5f_\mu - Q_\mu) \alpha_{-2}^\mu \right] |f\rangle. \quad (3.11)$$

The DGS corresponding to $\psi_{\frac{3}{2}, \pm\frac{1}{2}}^-$ are $G_{\frac{3}{2}, \pm\frac{1}{2}}^-$:

(type I)

$$G_{\frac{3}{2}, \pm\frac{1}{2}}^{-(1)} \sim \left[\left(\begin{pmatrix} \frac{5}{2} & \pm\frac{3}{2} \\ \pm\frac{3}{2} & \frac{1}{2} \end{pmatrix} \alpha_{-1}^\mu \alpha_{-1}^\nu + \left(\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \pm\frac{1}{\sqrt{2}} \end{pmatrix} \alpha_{-2}^\mu \right) \right] \left| f_\mu = \left(-\frac{5}{2}, \pm\frac{1}{2} \right) \right\rangle, \quad (3.12)$$

(type II)

$$G_{\frac{3}{2}, \pm\frac{1}{2}}^{-(2)} \sim \frac{1}{2} \left[\left(\begin{pmatrix} \frac{73}{2} & \pm\frac{15}{2} \\ \pm\frac{15}{2} & \frac{5}{2} \end{pmatrix} \alpha_{-1}^\mu \alpha_{-1}^\nu + \left(\begin{pmatrix} \frac{29}{\sqrt{2}} \\ \pm\frac{5}{\sqrt{2}} \end{pmatrix} \alpha_{-2}^\mu \right) \right] \left| f_\mu = \left(-\frac{5}{2}, \pm\frac{1}{2} \right) \right\rangle. \quad (3.13)$$

Note that a linear combination of these two states produces a pure ϕ DGS

$$G_{\frac{3}{2}, \pm\frac{1}{2}}^- \sim \left[(\partial\phi)^2 - \frac{1}{\sqrt{2}} \partial^2 \phi \right] e^{\pm \frac{i}{\sqrt{2}} X - \frac{5}{2} \phi} |0\rangle. \quad (3.14)$$

The DGS corresponding to discrete momenta of $\psi_{\frac{3}{2}, \pm \frac{1}{2}}^+$ are degenerate, i.e., the type I and type II DGS are linearly dependent:

$$G_{\frac{3}{2}, \pm \frac{1}{2}}^+ \sim \left[\begin{pmatrix} -\frac{1}{2} & \pm \frac{3}{2} \\ \pm \frac{3}{2} & -\frac{5}{2} \end{pmatrix} \alpha_{-1}^\mu \alpha_{-1}^\nu + \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \mp \frac{5}{\sqrt{2}} \end{pmatrix} \alpha_{-2}^\mu \right] \Big| f_\mu = \left(\frac{1}{2}, \pm \frac{1}{2} \right) \rangle \quad (3.15)$$

There is no pure ϕ DGS here. In general, the ψ^+ sector has fewer DGS than the ψ^- sector at the same discrete momenta, as a result, the pure ϕ DGS only arise at the minus sector. This fact is related to the degeneracy of the DGS in the plus sector. Historically the ψ^+ sector discrete states arise when one considers the singular gauge transformation constructed from the difference of the two plus gauge states [109–112, 114].

2. Generating the Discrete ZNS

In this section, we will give a general formula for the DGS. In general, there are many DGS for each discrete momentum. The higher the momentum is, the more numerous the DGS are found. We first express the discrete states in Eq.(3.7) in terms of Schur polynomials, which are defined as follows:

$$\text{Exp} \left(\sum_{k=1}^{\infty} a_k x^k \right) = \sum_{k=0}^{\infty} S_k(\{a_k\}) x^k \quad (3.16)$$

where S_k is the Schur polynomial, a function of $\{a_k\} = \{a_i : i \in \mathbb{Z}_k\}$. Performing the operator products in Eq.(3.7), the discrete states $\psi_{J,M}^\pm$ can be written as

$$\begin{aligned} \psi_{J,M}^\pm &\sim \prod_{i=1}^{J-M} \int \frac{dz_i}{2\pi i} z_i^{-2J} \prod_{j < k}^{J-M} (z_j - z_k)^2 \\ &\text{Exp} \left[\sum_{i=1}^{J-M} [-i\sqrt{2}X(z_i)] + \sqrt{2}(iJX(0) + (-1 \pm J)\phi(0)) \right]. \end{aligned} \quad (3.17)$$

We can write

$$\prod_{j < k}^{J-M} (z_j - z_k)^2 = \sum_f \begin{vmatrix} 1 & z_{f_1} & \cdots & z_{f_1}^{J-M-1} \\ z_{f_2} & z_{f_2}^2 & \cdots & z_{f_2}^{J-M-1} \\ \vdots & \vdots & \ddots & \vdots \\ z_{f_{J-M}}^{J-M-1} & z_{f_{J-M}}^{J-M} & \cdots & z_{f_{J-M}}^{2J-2M-2} \end{vmatrix}, \quad (3.18)$$

and Taylor expand $X(z_i)$ around $z_i = 0$

$$e^{-i\sqrt{2}X(z_i)} = e^{-i\sqrt{2}X(0)} \left[\sum_{k=0}^{\infty} S_k \left(\left\{ \frac{-i\sqrt{2}}{k!} \partial^k X(0) \right\} \right) z_i^k \right]. \quad (3.19)$$

In Eq.(3.18) the sum is over all permutations $f = (f_1, \dots, f_{J-M})$ of $(1, 2, \dots, J-M)$. Putting Eq.(3.18) and Eq.(3.19) into Eq.(3.17), and using the symmetry of the integrand over the index i , we have

$$\psi_{J,M}^{\pm} \sim \begin{vmatrix} S_{2J-1} & S_{2J-2} & \cdots & S_{J+M} \\ S_{2J-2} & S_{2J-3} & \cdots & S_{J+M-1} \\ \vdots & \vdots & \ddots & \vdots \\ S_{J+M} & S_{J+M-1} & \cdots & S_{2M+1} \end{vmatrix} \text{Exp} \left[\sqrt{2}(iMX(0) + (-1 \pm J)\phi(0)) \right] \quad (3.20)$$

with $S_k = S_k \left(\left\{ \frac{-i\sqrt{2}}{k!} \partial^k X(0) \right\} \right)$ and $S_k = 0$ if $k < 0$. We will denote the rank $(J-M)$ determinant in Eq.(3.20) as $\Delta(J, M, -i\sqrt{2}X)$. As a by-product, comparing the two definitions in Eq.(3.7) we can use Eq.(3.20) to deduce a mathematical identity relating the determinants of rank $(J-M)$ and $(J+M)$,

$$\Delta(J, M, -i\sqrt{2}X) = (-1)^{J+M+1} \Delta(J, -M, i\sqrt{2}X). \quad (3.21)$$

We now begin to study the DGS. One first notes that the DGS in Eq.(3.14) can be generated by $\int \frac{dz}{2\pi i} e^{-\sqrt{2}\phi(z)} \psi_{\frac{1}{2}, \pm \frac{1}{2}}^{-}(0)$. In general it is also possible to write down explicitly one of the many ZNS for each discrete momentum in the ψ^{-} sector as follows

$$\begin{aligned} G_{J,M}^{-} &\sim \left[\int \frac{dz}{2\pi i} e^{-\sqrt{2}\phi(z)} \right] \psi_{J-1,M}^{-} \\ &\sim S_{2J-1} \left(\left\{ \frac{-\sqrt{2}}{k!} \partial^k \phi \right\} \right) \Delta(J-1, M, -i\sqrt{2}X) e^{iMX + (-1-J)\phi}. \end{aligned} \quad (3.22)$$

Using Eq.(3.2), it can be verified explicitly after a length algebra that they are primary, and are of dimension 1. For $M = J-1$ Eq.(3.22) are pure ϕ states, but orthogonal to the pure X discrete physical states at the same momenta, and are therefore ZNS. For general M the polynomial prefactor in Eq.(3.22) factorizes into pure ϕ and pure X parts, and are still orthogonal to the physical states at the same momenta. They are, therefore, also ZNS. This is also suggested by the following result [17–19]

$$\int \frac{dz}{2\pi i} \psi_{J_1, M_1}^- \psi_{J_2, M_2}^- \sim 0 \quad (3.23)$$

where the r.h.s. is meant to be a DGS. We thus have explicitly obtained a DGS for each ψ^- discrete momentum. We stress that there are still other DGS in this sector, for example, the states

$$G'_{J, M} \sim \left[\int \frac{dz}{2\pi i} e^{-\sqrt{2}\psi(z)} \right]^{J-M} \psi_{M, M}^- \quad (3.24)$$

can be shown to be of dimension 1. Since they are pure ϕ states, they are also DGS. This expression reminds us of Eq.(3.7). However, there is no $SU(2)$ structure in the ϕ direction, and the usual techniques of $c = 1$ 2d conformal field theory cannot be applied. The pure ϕ DGS are only found in the minus sector.

For the plus sector, the operator products of the discrete states defined in Eq.(3.7) form a w_∞ algebra [17–19]

$$\int \frac{dz}{2\pi i} \psi_{J_1, M_1}^+ \psi_{J_2, M_2}^+ = (J_2 M_1 - J_1 M_2) \psi_{J_1+J_2-1, M_1+M_2}^+. \quad (3.25)$$

(Again, the r.h.s. is up to a DGS.) We can subtract two positive norm discrete states to obtain a pure gauge state as following

$$\begin{aligned} G_{J, M}^+ &= (J + M + 1)^{-1} \int \frac{dz}{2\pi i} [\psi_{1, -1}^+(z) \psi_{J, M+1}^+(0) + \psi_{J, M+1}^+(z) \psi_{1, -1}^+(0)] \\ &\sim (J - M)! \Delta(J, M, -i\sqrt{2}X) \text{Exp} \left[\sqrt{2}(iMX + (J - 1)\phi) \right] \\ &\quad + (-1)^{2J} \sum_{j=1}^{J-M} (J - M - 1)! \int \frac{dz}{2\pi i} \mathcal{D}(J, M, -i\sqrt{2}X(z), j) \\ &\quad \cdot \text{Exp} \left[\sqrt{2}(i(M + 1)X(z) + (J - 1)\phi(z) - X(0)) \right] \end{aligned} \quad (3.26)$$

where $\mathcal{D}(J, M, -i\sqrt{2}X(z), j)$ is defined as

$$\mathcal{D}(J, M, -i\sqrt{2}X(z), j) = \begin{vmatrix} S_{2J-1} & S_{2J-2} & \cdots & \cdots & S_{J+M} \\ S_{2J-2} & S_{2J-3} & \cdots & \cdots & S_{J+M-1} \\ \vdots & \vdots & \ddots & & \vdots \\ (-z)^{j-1-2J} & (-z)^{j-2J} & & & (-z)^{j-J-M-2} \\ \vdots & \vdots & & \ddots & \vdots \\ S_{J+M} & S_{J+M-1} & \cdots & \cdots & S_{2M+1} \end{vmatrix}, \quad (3.27)$$

which is the same as $\Delta(J, M, -i\sqrt{2}X(z))$ except that the j^{th} row is replaced by $\{(-z)^{j-1-2J}, (-z)^{j-2J} \dots\}$. As an example, with Eq.(3.26) one easily obtains the state $G_{\frac{3}{2}, \pm\frac{1}{2}}^+$ of Eq.(3.15).

3. w_∞ Charges and conclusion

It was shown [17–19] that the operators products of the states $\psi_{J,M}^+$ defined in Eq.(3.7) satisfy the w_∞ algebra in Eq.(3.25). By construction Eq.(3.26) one can easily see that the plus sector DGS $G_{J,M}^+$ carry the w_∞ charges and can be considered as the symmetry parameters of the theory. In fact, the operator products of the DGS $G_{J,M}^+$ of Eq.(3.26) form the same w_∞ algebra

$$\int \frac{dz}{2\pi i} G_{J_1, M_1}^+(z) G_{J_2, M_2}^+(0) = (J_2 M_1 - J_1 M_2) G_{J_1+J_2-1, M_1+M_2}^+(0) \quad (3.28)$$

where the r.h.s. is defined up to another DGS. The high energy limit of Eq.(3.26) will be discussed in section V.E of part II of this review.

In summary, we have shown that the spacetime w_∞ symmetry parameters of 2D string theory come from solution of equations Eq.(3.8) and Eq.(3.9). This argument is valid also in the case of 26D (or 10D) string theory although it would be very difficult to exhaust all the solutions of the ZNS [11–13]. This difficulty is, of course, related to the high dimensionality of spacetime. The DGS we introduced in the old covariant quantization in this chapter seem to be related to the ghost sectors and the ground ring structure [20, 21] in the BRST quantization of the theory.

B. 2D superstring theory

In this section, we will generalize our results in the previous section to $N = 1$ super-Liouville theory in the worldsheet supersymmetric way [23]. We will work out the DGS of the Neveu-Schwarz sector in the zero ghost picture. We first discuss the $N = 1$ super-Liouville theory and set up the notations. We then calculate the general formula for discrete positive-norm states in a worldsheet superfield form. Finally a general formula of discrete ZNS or DGS will be presented and w_∞ charges will then be calculated.

1. 2D super-Liouville theory

The $N = 1$ two dimensional supersymmetric Liouville action is given by [115]

$$S = \frac{1}{8\pi} \int d^2\mathbf{z} [g^{\alpha\beta} (\partial_\alpha \mathbf{X} \partial_\beta \mathbf{X} + \partial_\alpha \Phi \partial_\beta \Phi) - Q \hat{\mathbf{Y}} \Phi] \quad (3.29)$$

where Φ is the super-Liouville field, $\hat{\mathbf{Y}}$ the superfield curvature, $d\mathbf{z} = dzd\theta$ and with $\mathbf{X}^\mu = \begin{pmatrix} \Phi \\ \mathbf{X} \end{pmatrix}$,

$$\mathbf{X}^\mu(z, \theta, \bar{z}, \bar{\theta}) = X^\mu + \theta \psi^\mu + \bar{\theta} \bar{\psi}^\mu + \theta \bar{\theta} F^\mu. \quad (3.30)$$

Bold faced variables denote superfields hereafter.

For $\hat{c} = 1 = \frac{2}{3}c$ theory Q , which represents the background charge of the super-Liouville field, is set to be 2 so that the total conformal anomaly cancels that from conformal and superconformal ghost contribution.

The equations of motion show that the left and right-moving components of \mathbf{X}^μ decouple, and the auxiliary fields F^μ vanish. As a result, we need to consider only one of the chiral sectors, while the other (anti-holomorphic) sector has a similar formula. The stress energy tensor is

$$\mathbf{T}_{zz} = -\frac{1}{2} \mathbf{D} \mathbf{X}^\mu \mathbf{D}^2 \mathbf{X}_\mu - \frac{1}{2} Q \mathbf{D}^3 \Phi = T_F + \theta T_B \quad (3.31)$$

with

$$\begin{aligned} T_F &= -\frac{1}{2} \partial X^\mu \partial X_\mu - \frac{1}{2} Q \partial^2 X^0 + \frac{1}{2} \psi^\mu \partial \psi_\mu, \\ T_B &= -\frac{1}{2} \psi^\mu \partial X_\mu - \frac{1}{2} Q \partial \psi_0 \end{aligned} \quad (3.32)$$

where $\mathbf{D} = \partial_\theta + \theta \partial_z$, and now $\mathbf{X}^\mu = X^\mu(z) + \theta \psi^\mu(z)$.

For the Neveu-Schwarz sector, if we define the mode expansion by

$$\partial_z X^\mu = - \sum_{n=-\infty}^{\infty} z^{-n-1} (\alpha_n^0, i\alpha_n^1), \quad (3.33)$$

$$\psi^\mu = - \sum_{r \in \mathbb{Z} + \frac{1}{2}} z^{-r-\frac{1}{2}} (b_r^0, i b_r^1), \quad (3.34)$$

then we have

$$[\alpha_m^\mu, \alpha_n^\nu] = n \eta^{\mu\nu} \delta_{m+n}, \quad \{b_r^\mu, b_s^\nu\} = \eta^{\mu\nu} \delta_{r+s}. \quad (3.35)$$

With the Minkowski metric $\eta^{\mu\nu} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, $Q^\mu = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$ and the zero modes $\alpha_0^\mu = f^\mu = \begin{pmatrix} \epsilon \\ p \end{pmatrix}$, we find the super-Virasoro generators as modes of T_F and T_B ,

$$\begin{aligned}
L_n &= \left(\frac{n+1}{2} Q^\mu + f^\mu \right) \alpha_{\mu,n} + \frac{1}{2} \sum_{k \neq 0} : \alpha_{\mu,-k} \alpha_{n+k}^\mu : + \frac{1}{2} \sum_{r \in \mathbb{Z} + \frac{1}{2}} \left(r + n + \frac{1}{2} \right) : b_{-r}^\mu b_{n+r,\mu} :, \\
L_0 &= \frac{1}{2} (Q^\mu + f^\mu) f_\mu + \sum_{k=1}^{\infty} : \alpha_{\mu,-k} \alpha_k^\mu : + \frac{1}{2} \sum_{r \in \mathbb{Z} + \frac{1}{2}} \left(r + \frac{1}{2} \right) : b_{-r}^\mu b_{r,\mu} :, \\
G_r &= \sum_{s \in \mathbb{Z} + \frac{1}{2}} \alpha_{r-s}^\mu b_{\mu,s} + \left(r + \frac{1}{2} \right) Q^\mu b_{\mu,r}.
\end{aligned} \tag{3.36}$$

The vacuum $|0\rangle$ is annihilated by all α_n^μ and b_r^μ with $n > 0$ and $r > 0$. In the old covariant quantization of the theory, physical states $|\psi\rangle$ are those satisfying the conditions

$$\begin{aligned}
G_{\frac{1}{2}} |\psi\rangle &= G_{\frac{3}{2}} |\psi\rangle = 0 \\
\text{and} \quad L_0 |\psi\rangle &= \frac{1}{2} |\psi\rangle.
\end{aligned} \tag{3.37}$$

2. World-sheet superfield form of the discrete states

With Eq.(3.31) one can easily check that the two branches of massless “tachyon”

$$T^\pm(p) = \int d\mathbf{z} e^{ip\mathbf{X} + (\pm|p|-1)\Phi} \tag{3.38}$$

are positive norm physical states. It was also known that there exists discrete momentum physical states. Writing $\int d\mathbf{z} \Psi_{J,\pm J}^{(\pm)} = T^{(\pm)}(\pm J)$, the discrete states in the “material gauge” are

$$\Psi_{J,M}^{(\pm)} \sim (H^-)^{J-M} \Psi_{J,J}^{(\pm)} \sim (H^+)^{J+M} \Psi_{J,-J}^{(\pm)} \tag{3.39}$$

where

$$H^\pm = \sqrt{2} \int d\mathbf{z} e^{\pm i\mathbf{X}(\mathbf{z})}, \quad H^0 = \int d\mathbf{z} D\mathbf{X} \tag{3.40}$$

are zero modes of the level 2 $SU(2)_{\kappa=2}$ Kac-Moody algebra in $\hat{c} = 1$ $2d$ superconformal field theory. Here we note that the NS sector corresponds to states with $J \in \mathbb{Z}$ while the Ramond sector corresponds to those with $J \in \mathbb{Z} + \frac{1}{2}$.

To find the explicit expressions for the discrete states, we first define the super-Schur polynomials,

$$\mathbf{S}_k(-i\mathbf{X}) = \frac{\mathbf{D}^k e^{-i\mathbf{X}}}{[k/2]!} e^{i\mathbf{X}} \quad (3.41)$$

where $[\frac{k}{2}]$ denotes the integral part of $\frac{k}{2}$, as the $N = 1$ generalization to the Schur polynomial S_k , which is defined as

$$\text{Exp} \left(\sum_{k=1}^{\infty} a_k x^k \right) = \sum_{k=0}^{\infty} S_k(\{a_k\}) x^k. \quad (3.42)$$

Note that $S_k(\{i\partial^m \mathbf{X}/m!\}) = \mathbf{S}_{2k}(i\mathbf{X})$. Direct integration shows that

$$\begin{aligned} \int d\mathbf{z}_1 \frac{1}{(z_1 - z - \theta_1 \theta)^n} f(\mathbf{X}_1) &= \frac{\mathbf{D}^{2n-1} f(\mathbf{X})}{(n-1)!} \\ &= \frac{\partial_z^{2n-1} (f'_z f(X))}{(n-1)!}. \end{aligned} \quad (3.43)$$

Using Eq.(3.43) we obtain

$$\begin{aligned} \Psi_{J,J-1}^{\pm} &\sim \mathbf{S}_{2J-1}(-i\mathbf{X}) e^{i(J-1)\mathbf{X} + (\pm J-1)\Phi} \\ &= \frac{1}{(J-1)!} [-i\partial^{J-1} (e^{-iX^1} \psi^1) + \theta \partial^{J-1} e^{-iX^1}] e^{iJ\mathbf{X} + (\pm J-1)\Phi}. \end{aligned} \quad (3.44)$$

For example, by

$$(-\mathbf{D}^{2r} \Phi^0, i\mathbf{D}^{2r} \mathbf{X}^1) \rightarrow b_{-r}^{\mu}, \quad (-\mathbf{D}^{2n} \mathbf{X}^0, i\mathbf{D}^{2n} \mathbf{X}^1) \rightarrow \alpha_{-n}^{\mu}, \quad (3.45)$$

we have

$$\Psi_{1,0}^+ = \mathbf{D}\mathbf{X} \rightarrow b_{\frac{1}{2}}^1 |f^{\mu} = (0,0)\rangle \quad (3.46)$$

and

$$\begin{aligned} \Psi_{2,\pm 1}^+ &= [-i\mathbf{D}^3 \mathbf{X} - \mathbf{D}\mathbf{X}\mathbf{D}^2 \mathbf{X}] e^{\pm i\mathbf{X} + \Phi} \\ &\rightarrow [-b_{-\frac{3}{2}}^1 + b_{-\frac{1}{2}}^1 \alpha_{-1}^1] |f^{\mu} = (1, \pm 1)\rangle. \end{aligned} \quad (3.47)$$

They can be checked to satisfy the physical state conditions in Eq.(3.37).

Performing the operator products in Eq.(3.39), the discrete states $\Psi_{J,M}^{\pm}$ are

$$\begin{aligned} \Psi_{J,M}^{\pm} &\sim \prod_{i=1}^{J-M} \int d\mathbf{z}_i \mathbf{z}_{i0}^{-J} \prod_{j < k}^{J-M} \mathbf{z}_{jk} \\ &\text{Exp} \left[\sum_{i=1}^{J-M} [-i\mathbf{X}(\mathbf{z}_i)] + (iJ\mathbf{X}(\mathbf{z}_0) + (1 \pm J)\Phi(\mathbf{z}_0)) \right] \end{aligned} \quad (3.48)$$

where $\mathbf{z}_{ab} = z_a - z_b - \theta_a \theta_b$. If we write $\mathbf{z}_{ab} = \mathbf{z}_{a0} - \mathbf{z}_{b0} - (\theta_a - \theta_0)(\theta_b - \theta_0)$, and use $\int d\mathbf{z}_a (\theta_a - \theta_0) \mathbf{z}_{a0}^{-n} f(\mathbf{X}_a) = \mathbf{D}^{2n-2} f(\mathbf{X}_0) / (n-1)!$, we get, for $M = J - 2$,

$$\Psi_{J,J-2}^{\pm} \sim [2\mathbf{S}_{2J-3}\mathbf{S}_{2J-1} + \mathbf{S}_{2J-2}\mathbf{S}_{2J-2}]e^{i(J-2)\mathbf{X}+(\pm J-1)\Phi}. \quad (3.49)$$

The vertex operators correspond to the upper components of Eq.(3.49), i.e.,

$$\begin{aligned} \int d\theta \Psi_{J,J-2}^{\pm} \sim & [(iJ\psi^1 + (\pm J - 1)\psi^0)(S_{J-1}^2 + 2S_{J-\frac{3}{2}}^{NS}S_{J-\frac{1}{2}}^{NS}) \\ & - 2J(S_J S_{J-\frac{3}{2}}^{NS} - S_{J-1} S_{J-\frac{1}{2}}^{NS})] e^{i(J-2)X^1+(\pm J-1)X^0} \end{aligned} \quad (3.50)$$

where $S_J = S_J(\{-i\partial^m \mathbf{X}/m!\})$ and $S_{k+\frac{1}{2}}^{NS} = \sum_{m=0}^k \frac{-iS_m \partial^{k-m} \psi^1}{(k-m)!}$. Using Eq.(3.45) and Eq.(3.49) it is found that

$$\Psi_{2,0}^+ \rightarrow [2b_{-\frac{1}{2}}^1 b_{-\frac{3}{2}}^1 + \alpha_{-1}^1 \alpha_{-1}^1] |f^\mu = (1, 0)\rangle. \quad (3.51)$$

It can be checked that it satisfies the physical state conditions Eq.(3.37)

For $M = J - 3$, a straightforward calculation gives

$$\begin{aligned} \Psi_{J,J-3}^{\pm} \sim & [3!\mathbf{S}_{2J-1}\mathbf{S}_{2J-3}\mathbf{S}_{2J-5} + 3!\mathbf{S}_{2J-2}\mathbf{S}_{2J-3}\mathbf{S}_{2J-4} \\ & - \frac{3!}{1!2!}\mathbf{S}_{2J-1}\mathbf{S}_{2J-4}^2 - \frac{3!}{2!1!}\mathbf{S}_{2J-2}^2\mathbf{S}_{2J-5}] e^{i(J-3)\mathbf{X}+(\pm J-1)\Phi}. \end{aligned} \quad (3.52)$$

It is now easy to write down an expression for general M ,

$$\Psi_{J,M}^{\pm} \sim \left| \begin{array}{cccc} \mathbf{S}_{2J-1} & \mathbf{S}_{2J-2} & \cdots & \mathbf{S}_{J+M} \\ \mathbf{S}_{2J-2} & \mathbf{S}_{2J-3} & \cdots & \mathbf{S}_{J+M-1} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{S}_{J+M} & \mathbf{S}_{J+M-1} & \cdots & \mathbf{S}_{2M+1} \end{array} \right|' \text{Exp}[(iM\mathbf{X}(\mathbf{z}_0) + (-1 \pm J)\Phi(\mathbf{z}_0))] \quad (3.53)$$

with $\mathbf{S}_k = \mathbf{S}_k(-i\mathbf{X}(\mathbf{z}_0))$ and $\mathbf{S}_k = 0$ if $k < 0$. We will denote the rank $(J - M)$ “primed”-determinant in Eq.(3.53) as $\Delta'(J, M, -i\mathbf{X})$, which (by definition) has all the signed terms in the normal determinant, except with a multiplicity of the multinomial coefficient $\frac{(J-M)!}{n_a!n_b!\dots}$ for the term $\mathbf{S}_a^{n_a}\mathbf{S}_b^{n_b}\dots$ (where $\sum_a n_a = J - M$).

3. Discrete ZNS and w_∞ charges

It was known [116–118] that the discrete states in Eq.(3.39) satisfy the w_∞ algebra

$$\int d\mathbf{z} \Psi_{J_1, M_1}^+(\mathbf{z}) \Psi_{J_2, M_2}^+(\mathbf{0}) = (J_2 M_1 - J_1 M_2) \Psi_{J_1+J_2-1, M_1+M_2}^+(\mathbf{0}), \quad (3.54)$$

$$\int d\mathbf{z} \Psi_{J_1, M_1}^-(\mathbf{z}) \Psi_{J_2, M_2}^-(\mathbf{0}) \sim 0 \quad (3.55)$$

where the RHS is defined up to a DGS.

In general, there are two types of ZNS in the old covariant quantization of the theory,

Type I:

$$|\psi\rangle = G_{-\frac{1}{2}} |\chi\rangle \quad \text{where} \quad G_{\frac{1}{2}} |\chi\rangle = G_{\frac{3}{2}} |\chi\rangle = L_0 |\tilde{\chi}\rangle = 0. \quad (3.56)$$

Type II:

$$|\psi\rangle = \left(G_{-\frac{3}{2}} + 2L_{-1} G_{-\frac{1}{2}} \right) |\tilde{\chi}\rangle \quad \text{where} \quad G_{\frac{1}{2}} |\tilde{\chi}\rangle = G_{\frac{3}{2}} |\tilde{\chi}\rangle = 0 \\ (L_0 + 1) |\tilde{\chi}\rangle = 0. \quad (3.57)$$

They satisfy the physical state conditions Eq.(3.37), and have zero norm. There is an infinite number of continuum momentum ZNS solutions for Eq.(3.56) and Eq.(3.57). However, as far as the dynamics is concerned, we are only interested in those with discrete momentum.

At mass level one, $f_\mu(f^\mu + Q^\mu) = 0$, only ZNS of type I are found: $f_\mu \alpha_{-1}^\mu |f\rangle$, where $|f\rangle =: e^{ip\mathbf{X} + \epsilon\Phi} : |0\rangle$. The DGS $G_{1,0}^- =: \mathbf{D}\Phi e^{-2\Phi} : |0\rangle$ corresponds to the momentum of $\Psi_{1,0}^-$. There is no corresponding DGS for $\Psi_{1,0}^+ =: \mathbf{D}\mathbf{X} :$.

At the next mass level, $f_\mu(f^\mu + Q^\mu) = -2$, $N_{\mu\nu} = -N_{\nu\mu}$ and $M_\mu = 2N_{\mu\nu}(f^\nu + Q^\nu)$, the type I ZNS is found to be

$$|\psi\rangle = [(M_\mu f_\nu \alpha_{-1}^\mu b_{-\frac{1}{2}}^\nu + M_\mu b_{-\frac{3}{2}}^\nu + 2N_{\mu\nu} \alpha_{-1}^\mu b_{-\frac{1}{2}}^\nu] |f\rangle, \quad (3.58)$$

while the type II state is

$$|\psi\rangle = [(2f_\mu f_\nu + \eta_{\mu\nu}) \alpha_{-1}^\mu b_{-\frac{1}{2}}^\nu + (3f_\mu - Q_\mu) b_{-\frac{3}{2}}^\mu] |f\rangle. \quad (3.59)$$

As in the bosonic Liouville theory [22], the ZNS corresponding to the discrete momenta of $\Psi_{2,\pm 1}^+$ are degenerate, i.e., the type I and type II gauge states are linearly dependent:

$$G_{2,\pm 1}^+ \sim \left[\begin{pmatrix} 1 & \mp 2 \\ \mp 2 & 3 \end{pmatrix} \alpha_{-1}^\mu b_{-\frac{1}{2}}^\nu + \begin{pmatrix} -1 \\ \pm 3 \end{pmatrix} b_{-\frac{3}{2}}^\mu \right] |f^\mu = (1, \pm 1)\rangle. \quad (3.60)$$

For the minus sector, type I DGS is

$$G_{2,\pm 1}^{-,I} \sim \left[\begin{pmatrix} 3 & \pm 2 \\ \pm 2 & 1 \end{pmatrix} \alpha_{-1}^\mu b_{-\frac{1}{2}}^\nu + \begin{pmatrix} 1 \\ \pm 1 \end{pmatrix} b_{-\frac{3}{2}}^\mu \right] |f^\mu = (-3, \pm 1)\rangle, \quad (3.61)$$

and type II DGS is

$$G_{2,\pm 1}^{-,II} \sim \left[\begin{pmatrix} 17 & \pm 6 \\ \pm 6 & 3 \end{pmatrix} \alpha_{-1}^\mu b_{-\frac{1}{2}}^\nu + \begin{pmatrix} 11 \\ \pm 3 \end{pmatrix} b_{-\frac{3}{2}}^\mu \right] |f^\mu = (-3, \pm 1)\rangle. \quad (3.62)$$

Note that $3G_{2,\pm 1}^{-,I} - G_{2,\pm 1}^{-,II}$ is a “pure Φ ” DGS, similar to the DGS in the bosonic Liouville theory.

We now apply the scheme used in [22] to derive a general formula for the DGS. From Eq.(3.55), the DGS in the minus sector can be written down explicitly as follows

$$\begin{aligned} \mathbf{G}_{J,M}^- &\sim \left[\int d\mathbf{z} e^{-\Phi(\mathbf{z})} \right] \Psi_{J-1,M}^-(\mathbf{0}) \\ &\sim \mathbf{S}_{2J-1}(-\Phi) \Delta^{[iM\mathbf{X}+(-1-J)\Phi]}. \end{aligned} \quad (3.63)$$

We thus have explicitly obtained a DGS for each Ψ^- discrete momentum. However, there are still other DGS in this sector, for example, the states

$$\mathbf{G}_{J,M}'^- \sim \left[\int d\mathbf{z} e^{-\Phi(\mathbf{z})} \right]^{J-M} \Psi_{M,M}^-(\mathbf{0}) \quad (3.64)$$

can be shown to satisfy the physical state conditions. Since they are “pure Φ ” states, they are also DGS. For example, $\mathbf{G}_{1,0}^- = \mathbf{D}\Phi e^{-\Phi}$ and $\mathbf{G}_{2,\pm 1}^- = [-\mathbf{D}^3\Phi + \mathbf{D}\Phi\mathbf{D}^2\Phi]e^{\pm i\mathbf{X}-3\Phi}$, which is a linear combination of Eq.(3.61) and Eq.(3.62).

For the plus sector, we can subtract two (distinct) positive norm discrete states at the same momentum to obtain a pure DGS

$$\mathbf{G}_{J,M}^+ = (J+M+1)^{-1} \int d\mathbf{z} [\Psi_{1,-1}^+(\mathbf{z})\Psi_{J,M+1}^+(\mathbf{0}) - \Psi_{J,M+1}^+(\mathbf{z})\Psi_{1,-1}^+(\mathbf{0})]. \quad (3.65)$$

As an example, with Eq.(3.65) one finds

$$\begin{aligned} \mathbf{G}_{2,\pm 1}^+ &= [\pm 3i\mathbf{D}^3\mathbf{X} + \mathbf{D}^3\Phi + 3i\mathbf{D}^2\mathbf{X}\mathbf{D}\mathbf{X} \\ &\quad \pm 2i\mathbf{D}^2\mathbf{X}\mathbf{D}\Phi \pm 2i\mathbf{D}\mathbf{X}\mathbf{D}^2\Phi + \mathbf{D}\Phi\mathbf{D}^2\Phi]e^{\pm i\mathbf{X}+\Phi} \end{aligned} \quad (3.66)$$

which is exactly the state we found in Eq.(3.60). We thus have explicitly obtained a DGS for each Ψ^+ momentum.

By construction in Eq.(3.65) one can see that $\mathbf{G}_{J,M}^+$ carry the w_∞ charges and serve as the symmetry parameters of the theory. In fact, their operator products form the same w_∞ algebra

$$\int d\mathbf{z} \mathbf{G}_{J_1,M_1}^+(\mathbf{z})\mathbf{G}_{J_2,M_2}^+(\mathbf{0}) = (J_2M_1 - J_1M_2)\mathbf{G}_{J_1+J_2-1,M_1+M_2}^+(\mathbf{0}) \quad (3.67)$$

where the RHS is defined up to another DGS.

We have demonstrated that the spacetime w_∞ symmetry parameters in the $2D$ superstring theory come from solution of equations Eq.(3.56) and Eq.(3.57). This phenomenon should survive in the more realistic high dimensional string theory [11, 12, 14], although it would be difficult to find the general solution of ZNS (due to the high dimensionality of spacetime). The DGS in the old covariant quantization of the theory is related to the ground ring structure in the BRST approach.

IV. SOLITON ZNS IN COMPACT SPACE AND ENHANCED GAUGE SYMMETRY

In this chapter we calculate ZNS in the spectrum of compactified closed and open string theories [24, 25]. For simplicity we will only do the calculations on torus compactifications. The programs can certainly be generalized to more complicated background geometries. We will see that there exist soliton ZNS which generate various enhanced stringy symmetries of the theories.

A. Compactified closed string

In this section, we study soliton gauge states in the spectrum of bosonic string compactified on torus. The enhanced Kac-Moody gauge symmetry, and thus T-duality, is shown to be related to the existence of these soliton ZNS in some moduli points.

1. Soliton ZNS on $R^{25} \otimes T^1$

In the simplest torus compactification, one coordinate of the string was compactified on a circle of radius R

$$X^{25}(\sigma + 2\pi, \pi) = X^{25}(\sigma, \pi) + 2\pi R_n \quad (4.1)$$

The single valued condition of the wave function then restricts the allowed momenta to be $p^{25} = m/R$ with $m, n \in \mathbb{Z}$. The mode expansion of the compactified coordinate for right

(left) mover is

$$X_R^{25} = \frac{1}{2}x^{25} + \left(p^{25} - \frac{1}{2}nR\right)(\tau - \sigma) + i \sum_{r \neq 0} \frac{1}{r} \alpha_r^{25} e^{-ir(\tau - \sigma)}, \quad (4.2)$$

$$X_L^{25} = \frac{1}{2}x^{25} + \left(p^{25} + \frac{1}{2}nR\right)(\tau + \sigma) + i \sum_{r \neq 0} \frac{1}{r} \tilde{\alpha}_r^{25} e^{-ir(\tau + \sigma)}. \quad (4.3)$$

We have normalized the string tension to be $\frac{1}{4\pi T} = 1$ or $\alpha' = 2$. The Virasoro operators can be written as

$$L_0 = \frac{1}{2} \left(p^{25} - \frac{1}{2}nR\right) + \frac{1}{2}p^{\mu^2} + \sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n, \quad (4.4)$$

$$\tilde{L}_0 = \frac{1}{2} \left(p^{25} + \frac{1}{2}nR\right) + \frac{1}{2}p^{\mu^2} + \sum_{n=1}^{\infty} \tilde{\alpha}_{-n} \cdot \tilde{\alpha}_n, \quad (4.5)$$

and

$$L_m = \frac{1}{2}\alpha_0^2 + \sum_{n=-\infty}^{\infty} \alpha_{m-n} \cdot \alpha_n, \quad (4.6)$$

$$\tilde{L}_m = \frac{1}{2}\tilde{\alpha}_0^2 + \sum_{n=-\infty}^{\infty} \tilde{\alpha}_{m-n} \cdot \tilde{\alpha}_n \quad (m \neq 0) \quad (4.7)$$

where

$$\alpha_0^{25} = p^{25} - \frac{1}{2}nR \equiv p_R^{25}, \quad (4.8)$$

$$\tilde{\alpha}_0^{25} = p^{25} + \frac{1}{2}nR \equiv p_L^{25}, \quad (4.9)$$

and the 25d momentum is $\alpha_0^\mu = \tilde{\alpha}_0^\mu = p^\mu \equiv k^\mu$. In the old covariant quantization of the theory, in addition to the physical propagating states, there are four types of ZNS in the spectrum

$$I.a \quad |\psi\rangle = L_{-1} |\chi\rangle \text{ where } L_m |\chi\rangle = 0, \left(\tilde{L}_m - \delta_m\right) |\chi\rangle = 0, \quad (m = 0, 1, 2, \dots), \quad (4.10)$$

$$II.a \quad |\psi\rangle = \left(L_{-2} + \frac{3}{2}L_{-1}^2\right) |\chi\rangle \text{ where } (L_m + \delta_m) |\chi\rangle = 0, \left(\tilde{L}_m - \delta_m\right) |\chi\rangle = 0, \quad (m = 0, 1, 2, \dots), \quad (4.11)$$

and by interchanging all left and right mover operators, one gets *I.b* and *II.b* states. Type II states are ZNS only at critical space-time dimension. We will only calculate type *a* states. Similar results can be easily obtained for type *b* states. For type *I.a* state, the $m = 0$ constraint of Eq.(4.10) gives

$$M^2 = \frac{m^2}{R^2} + \frac{1}{4}n^2R^2 + N + \tilde{N} - 1, \quad (4.12)$$

$$N - \tilde{N} = mn - 1 \quad (4.13)$$

where $N \equiv \sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n$ and $\tilde{N} \equiv \sum_{n=1}^{\infty} \tilde{\alpha}_{-n} \cdot \tilde{\alpha}_n$. For massless $M^2 = 0$ states, $N + \tilde{N} = 0$ or 1. The solutions of Eq.(4.12) and Eq.(4.13) are

$$N = 0, \tilde{N} = 1, m = n = 0 \text{ (any } R) \quad (4.14)$$

or

$$N = \tilde{N} = 0, m = n = \pm 1, R = \sqrt{2}. \quad (4.15)$$

Eq.(4.15) gives us our first soliton ZNS. It is easy to write down the explicit form of $|\chi\rangle$ and $|\psi\rangle$, and impose the $m \neq 0$ constraints of Eq.(4.10). There are also a vector and a scalar ZNS in Eq.(4.14). Similar results can be obtained for the type *I.b* state. In this case, $m = -n = \pm 1$. There is no type II solution in the massless case. We note that there are massless soliton ZNS only when $R = \sqrt{2}$ which is known as self-dual point in the moduli space. The vertex operators of all ZNS are calculated to be

$$k_\mu \theta_\nu \partial X_R^\mu \bar{\partial} X_L^\nu e^{ikx}; \quad L \leftrightarrow R, \quad (4.16)$$

$$k_\mu \partial X_R^\mu \bar{\partial} X_L^{25} e^{ikx}, \quad (4.17)$$

$$k_\mu \bar{\partial} X_L^\mu \partial X_R^{25} e^{ikx}, \quad (4.18)$$

$$k_\mu \partial X_R^\mu e^{\pm i\sqrt{2}X_L^{25}} e^{ikx}, \quad (4.19)$$

$$k_\mu \partial X_L^\mu e^{\pm i\sqrt{2}X_R^{25}} e^{ikx}. \quad (4.20)$$

It is easy to see that the three ZNS of Eq.(4.18) and (Eq.(4.20) form a representation of $SU(2)_R$ Kac-Moody algebra. Similarly, Eqs. Eq.(4.17) and Eq.(4.19) form a representation

of $SU(2)_L$ Kac-Moody algebra. The vector ZNS in Eq.(4.16) are responsible for the gauge symmetry of graviton and antisymmetric tensor field. We see that the self-dual point $R = \sqrt{2}$ is very special even from the gauge sector point of view.

2. Soliton ZNS on $R^{26-D} \otimes T^D$

In this section we compactify D coordinates on a D -dimensional torus $T^D \equiv \frac{R^D}{2\pi\Lambda^D}$

$$\vec{X}(\sigma + 2\pi, \pi) = \vec{X}(\sigma, \pi) + 2\pi\vec{L} \quad (4.21)$$

with

$$\vec{L} = \sum_{i=1}^D n_i \left(R_i \frac{\vec{e}_i}{\sqrt{2}} \right) \in (\Lambda^D) \quad (4.22)$$

where Λ^D is a D -dimensional lattice with a basis $\left\{ R_1 \frac{\vec{e}_1}{\sqrt{2}}, R_2 \frac{\vec{e}_2}{\sqrt{2}}, \dots, R_D \frac{\vec{e}_D}{\sqrt{2}} \right\}$. We have chosen $|\vec{e}_i|^2 = 2$. The allowed momenta \vec{p} take values on the dual lattice of Λ^D

$$\vec{p} = \sum_{i=1}^D m_i \left(\frac{1}{R_i} \sqrt{2} \vec{e}_i^* \right) \in (\Lambda^D)^*. \quad (4.23)$$

The basis of $(\Lambda^D)^*$ is $\left\{ \frac{1}{R_1} \sqrt{2} \vec{e}_1^*, \frac{1}{R_2} \sqrt{2} \vec{e}_2^*, \dots, \frac{1}{R_D} \sqrt{2} \vec{e}_D^* \right\}$ and we have $\vec{e}_i \cdot \vec{e}_i^* = \delta_{ij}$. The mode expansion of the compactified coordinates is

$$\vec{X}_R = \frac{1}{2} \vec{x} + \left(\vec{p} - \frac{1}{2} \vec{L} \right) (\tau - \sigma) + i \sum_{r \neq 0} \frac{1}{r} \alpha_r^{25} e^{-ir(\tau - \sigma)}, \quad (4.24)$$

$$\vec{X}_L = \frac{1}{2} \vec{x} + \left(\vec{p} + \frac{1}{2} \vec{L} \right) (\tau + \sigma) + i \sum_{r \neq 0} \frac{1}{r} \alpha_r^{25} e^{-ir(\tau + \sigma)}. \quad (4.25)$$

The right and left momenta are defined to be $\vec{p}_R = \left(\vec{p} - \frac{1}{2} \vec{L} \right)$ and $\vec{p}_L = \left(\vec{p} + \frac{1}{2} \vec{L} \right)$. It can be shown that the $2D$ -vector $\left(\vec{p}_R, \vec{p}_L \right)$ build an even self-dual Lorentzian lattice $\Gamma_{D,D}$, which guarantees the string one loop modular invariance of the theory [119, 120]. The moduli space of the theory is [121]

$$\mu = \frac{SO(D, D)}{SO(D) \times SO(D)} / O(D, D, Z) \quad (4.26)$$

where $O(D, D, Z)$ is the discrete T-duality group and $\dim \mu = D^2$. To complete the parametrization of the moduli space, one needs to introduce an antisymmetric tensor field B_{ij} in the bosonic string action. This will modify the right (left) momenta to be

$$\vec{p}_R = \left(\vec{p}_B - \frac{1}{2} \vec{L} \right), \quad (4.27)$$

$$\vec{p}_L = \left(\vec{p}_B + \frac{1}{2} \vec{L} \right) \quad (4.28)$$

where

$$\vec{p}_B = \sum_{i,j} \left(m_i \frac{1}{R_i} \sqrt{2} \vec{e}_i^* - n_j \frac{1}{\sqrt{2} R_i} B_{ij} \vec{e}_i^* \right). \quad (4.29)$$

We are now ready to discuss the gauge state. As a first step, we restrict ourselves to moduli space with $B_{ij} = 0$. For the type *I.a* state, the $m = 0$ constraint of Eq.(4.10) for massless states gives

$$N + \tilde{N} + \vec{p}^2 + \frac{1}{4} \vec{L}^2 = 1, \quad (4.30)$$

$$N - \tilde{N} = \sum_i m_i n_i - 1. \quad (4.31)$$

It is easy to see $N + \tilde{N} = 0$ or 1 . For $N + \tilde{N} = 1$, $m_i = n_i = 0$, we have trivial ZNS solutions. Soliton ZNS exists for the case $N + \tilde{N} = 0$ and the following moduli points

$$R_i = \sqrt{2}, e_i^I = \sqrt{2} \delta_i^I \quad (i = 1, 2, \dots, d) \quad (4.32)$$

with $m_i = n_i = \pm 1$, and $m_j = n_j = 0$ for $d < j \leq D$. In each case, the ZNS and soliton ZNS form a representation of $SU(2)^d$ algebra. Similar results can be easily obtained for the type *I.b* soliton ZNS. As in section II, there is no massless type II soliton ZNS. We now discuss $B_{ij} \neq 0$ case. For illustration, we choose $D = 2$. In this case $B_{ij} = B \epsilon_{ij}$, and one has four moduli parameters R_1, R_2, B , and $\vec{e}_1 \cdot \vec{e}_2$. For type *I.a* state, the $m = 0$ constraint of Eq.(4.9) gives

$$N + \tilde{N} + \vec{p}_B^2 + \frac{1}{4} \vec{L}^2 = 1, \quad (4.33)$$

$$N - \tilde{N} = m_1 n_1 + m_2 n_2 - 1. \quad (4.34)$$

soliton ZNS exists only for $N + \tilde{N} = 0$. For the moduli point

$$R_1 = R_2 = \sqrt{2}, B = \frac{1}{2}, \vec{e}_1 = (\sqrt{2}, 0), \vec{e}_2 = \left(-\sqrt{\frac{1}{2}}, \sqrt{\frac{3}{2}}\right), \quad (4.35)$$

one gets six soliton ZNS with momenta \vec{p}_R being the six root vectors of $SU(3)_R$. Together with two other trivial ZNS corresponding to $N = 0, \tilde{N} = 1$, they form the Frenkel-Kac-Segal [122] representation of $SU(3)_{k=1}$ Kac-Moody algebra. Note that \vec{e}_1, \vec{e}_2 are the two simple roots of $SU(3)$ and $\vec{e}_1^* = \left(\sqrt{\frac{1}{2}}, \sqrt{\frac{1}{6}}\right), \vec{e}_2^* = \left(0, \sqrt{\frac{2}{3}}\right)$. The six sets of winding number are $(m_1, n_1, m_2, n_2) = (1, 1, 0, 0), (-1, -1, 0, 0), (0, 0, 1, 1), (0, 0, -1, -1), (1, 1, 1, 0), (-1, -1, -1, 0)$. Similar results can be obtained for type *I.b* soliton ZNS. The ZNS (including soliton ZNS) thus form a representation of enhanced $SU(3)_R \otimes SU(3)_L$ at the moduli point of Eq.(4.35). In general, we expect that all enhanced Kac-Moody gauge symmetry at any moduli point should have a realization on soliton ZNS.

3. Massive soliton ZNS

In this section we derive the massive soliton ZNS at the first massive level $M^2 = 2$. We will find that soliton ZNS exists at infinite number of moduli points. One can also show that they exist at an infinite number of massive level. The existence of these massive soliton ZNS implies that there is an infinite enhanced gauge symmetry structure of compactified string theory. For type *I.a* state, the $m = 0$ constraint of Eq.(4.10) gives

$$\frac{m^2}{R^2} + \frac{1}{4}n^2 R^2 + N + \tilde{N} = 3, \quad (4.36)$$

$$N - \tilde{N} = mn - 1, \quad (4.37)$$

which implies $N + \tilde{N} = 0, 1, 2, 3$. Eq.(4.36) and Eq.(4.37) can be easily solved as following:

1. $N + \tilde{N} = 3$:

$$m = n = 0, N = 1, \tilde{N} = 2, \text{ any } R. \quad (4.38)$$

2. $N + \tilde{N} = 2$:

$$\begin{aligned} mn = 1, N = \tilde{N} = 1, R = \sqrt{2}, \\ mn = -1, N = 0, \tilde{N} = 2, R = \sqrt{2}. \end{aligned} \quad (4.39)$$

3. $N + \tilde{N} = 1$:

$$\begin{aligned} mn = 2, N = 1, \tilde{N} = 0, R = 2, 1. \text{ (} T - \text{duality) ,} \\ mn = 0, N = 0, \tilde{N} = 1, R = \frac{|m|}{\sqrt{2}}, \frac{2\sqrt{2}}{|m|}. \text{ (} T - \text{duality) .} \end{aligned} \quad (4.40)$$

4. $N + \tilde{N} = 0$:

$$mn = 1, N = \tilde{N} = 1, R = 2 \pm \sqrt{2}. \text{ (} T - \text{duality) } \quad (4.41)$$

where we have included a T-duality transformation $R \rightarrow \frac{2}{R}$ for some moduli points. Note that Eq.(4.40) tells us that massive soliton ZNS exists at an infinite number of moduli point. For type *II.a* state, the $m = 0$ constraint of Eq.(4.11) gives

$$\frac{m^2}{R^2} + \frac{1}{4}n^2R^2 + N + \tilde{N} = 2, \quad (4.42)$$

$$N - \tilde{N} = mn - 2, \quad (4.43)$$

which implies $N + \tilde{N} = 0, 1, 2$. Eq.(4.42) and Eq.(4.43) can be solved as following:

1. $N + \tilde{N} = 2$:

$$m = n = 0, N = 0, \tilde{N} = 2, \text{ any } R. \quad (4.44)$$

2. $N + \tilde{N} = 1$:

$$mn = 1, N = 0, \tilde{N} = 1, R = \sqrt{2}. \quad (4.45)$$

3. $N + \tilde{N} = 0$:

$$mn = 2, N = \tilde{N} = 0, R = 2, 1. \text{ (} T - \text{duality) .} \quad (4.46)$$

The vertex operators of all soliton ZNS can be easily calculated and written down. Similar results can be obtained for type b ZNS. One can also calculate propagating soliton states by using the same technique. We summarize the moduli points which exist soliton state and soliton ZNS as following:

a. Soliton ZNS :

$$R = \sqrt{2}, 2 \pm \sqrt{2}, \frac{|m|}{\sqrt{2}}, \frac{2\sqrt{2}}{|m|}, 2, 1. \quad (4.47)$$

b. *Soliton states* :

$$R = \sqrt{2}, 2 \pm \sqrt{2}, \frac{|m|}{\sqrt{2}}, \frac{2\sqrt{2}}{|m|}, \frac{|m|}{2}, \frac{4}{|m|}. \quad (4.48)$$

In Eq.(4.47) and Eq.(4.48), $m \in Z_+$. There is one interesting remark we would like to point out by the end of this section. One notes that in the second case of Eq.(4.40), instead of specifying $M^2 = 2$, in general we have

$$\frac{m^2}{R^2} + \frac{1}{4}n^2R^2 = M^2 \quad (4.49)$$

with $mn = 0$. For say $R = \sqrt{2}$, one gets $M^2 = \frac{m^2}{2}$ ($n = 0$). This means that we have an infinite number of massive soliton ZNS at any higher massive level of the spectrum. One can even explicitly write down the vertex operators of these soliton ZNS. We conjecture that the w_∞ symmetry of $2D$ string theory [22, 108] can be realized in these soliton ZNS. Other moduli points also consist of higher massive soliton ZNS in the spectrum.

Many known spacetime symmetries of string theory can be shown to be related to the existence of ZNS in the spectrum. The Heterotic ZNS for the $10D$ Heterotic string [11] and the discrete ZNS [22, 108] for the toy $2D$ string are such examples. We have introduced soliton ZNS for compactified closed string in this chapter, and have related them to the enhanced Kaluza-Klein Kac-Moody symmetries in the theory. In many cases, especially for the massive states, it is easier to study stringy symmetries in the ZNS sector than in the propagating spectrum directly.

Since the discrete T-duality symmetry group for bosonic closed string is the Weyl subgroup of the enhanced gauge group, it can also be considered as due to the existence of soliton

ZNS. It is not clear whether other discrete duality symmetry group can be understood in this way. Finally, it would be interesting to consider more complicated compactification, e.g. orbifold and Calabi-Yau compactifications and study the relation between soliton ZNS and duality symmetries.

B. Compactified open string

In this section, we study the mechanism of enhanced gauge symmetry of bosonic open string compactified on torus by analyzing the ZNS (nonzero winding of Wilson line) in the spectrum [25]. Unlike the closed string case discussed in the previous section, we will find that the soliton ZNS exist only at massive levels.

These soliton ZNS correspond to the existence of enhanced massive stringy symmetries with transformation parameters containing both Einstein and Yang-Mills indices in the case of Heterotic string [11]. In the T-dual picture, these symmetries exist only at some discrete values of compactified radii when N D -branes are coincident.

1. Chan-Paton ZNS

We first discuss ZNS of uncompactified open string with Chan-Paton factor and its implication on on-shell symmetry and Ward identity. For simplicity, we consider the oriented $U(N)$ case. The vertex operators of massless gauge state is

$$\theta^a \lambda_{ij}^a k \cdot \partial x e^{ikx} \quad (4.50)$$

where $\lambda \in U(N)$, $i \in N$, $j \in \bar{N}$ and $a \in$ adjoint representation of $U(N)$. The on-shell conformal deformation and $U(N)$ gauge symmetry to lowest order in the weak background field approximation are ($\square \theta^a = 0$, $\square \equiv \partial_\mu \partial^\mu$)

$$\delta T = \lambda_{ij}^a \partial_\mu \theta^a \partial x^\mu, \quad (4.51)$$

and

$$\delta A_\mu^a = \partial_\mu \theta^a \quad (4.52)$$

with T the energy momentum tensor and A_μ^a the massless gauge field.

One can verify the corresponding Ward identity by calculating e.g., 1-vector and 3-tachyons four point correlators. The amplitude is calculated to be

$$\begin{aligned} T_\mu^{abcd} &= \int \prod_{i=1}^4 dx_i \langle e^{ik_1 x_1} \partial_{x_\mu} e^{ik_2 x_2} e^{ik_3 x_3} e^{ik_4 x_4} \rangle T_r (\lambda^a \lambda^b \lambda^c \lambda^d) \\ &= \frac{\Gamma(-\frac{s}{2}-1) \Gamma(-\frac{t}{2}-1)}{\Gamma(\frac{u}{2}+1)} \left[k_{3\mu} \left(\frac{s}{2} + 1 \right) - k_{1\mu} \left(\frac{t}{2} + 1 \right) \right] \times T_r (\lambda^a \lambda^b \lambda^c \lambda^d) \end{aligned} \quad (4.53)$$

In Eq.(2.4), s, t and u are the usual Mandelstam variables. One can then verify the Ward identity

$$\theta^b k_2^\mu T_\mu^{abcd} = 0. \quad (4.54)$$

We now discuss the massive ZNS. The vertex operator of type I massive vector ZNS is

$$\theta_\mu^a \lambda_{ij}^a [k \cdot \partial x \partial x^\mu + \partial^2 x^\mu] e^{ikx}. \quad (4.55)$$

We note that the ZNS polarization contains both Einstein and Yang-Mills indices. This is very similar to the $10d$ closed Heterotic string case [11, 12]. The only difference is that in the Heterotic string, one could have more than one Yang-Mills index. The on-shell conformal deformation and the mixed Einstein-Yang-Mills-type symmetry to lowest order weak field approximation are $((\square - 2) \theta_\mu^a = \partial \cdot \theta^a = 0)$

$$\delta T = \lambda_{ij}^a \partial_\mu \theta_\nu^a \partial x^\mu \partial x^\nu + \lambda_{ij}^a \theta_\mu^a \partial^2 x^\mu \quad (4.56)$$

and

$$\delta M_{\mu\nu}^a = \partial_\mu \theta_\nu^a + \partial_\nu \theta_\mu^a. \quad (4.57)$$

One can also derive the corresponding massive Ward identity by calculating the decay rate of one massive state to three tachyons. The most general amplitude is calculated to be

$$A^{abcd} = \varepsilon^a \varepsilon^c \varepsilon^d (\varepsilon_{\mu\nu}^b T^{\mu\nu} + \varepsilon_\mu^b T^\mu) T_r (\lambda^a \lambda^b \lambda^c \lambda^d) \quad (4.58)$$

where

$$T^{\mu\nu} = \frac{\Gamma\left(-\frac{s}{2}-1\right)\Gamma\left(-\frac{t}{2}-1\right)}{\Gamma\left(\frac{u}{2}+2\right)} \left\{ \frac{s}{2}\left(\frac{s}{2}+1\right)k_3^\mu k_3^\nu + \frac{t}{2}\left(\frac{t}{2}+1\right)k_1^\mu k_1^\nu - 2\left(\frac{s}{2}+1\right)\left(\frac{t}{2}+1\right)k_1^\mu k_3^\nu \right\} \quad (4.59)$$

and

$$T^\mu = \frac{\Gamma\left(-\frac{s}{2}-1\right)\Gamma\left(-\frac{t}{2}-1\right)}{\Gamma\left(\frac{u}{2}+2\right)} \left\{ -k_3^\mu \frac{s}{2}\left(\frac{s}{2}+1\right) - k_1^\mu \frac{t}{2}\left(\frac{t}{2}+1\right) \right\}. \quad (4.60)$$

In Eq.(4.58) ε^a etc. are polarizations corresponding to tachyons and $(\varepsilon_{\mu\nu}^b, \varepsilon_\mu^b)$ is polarization of the massive state. The above amplitude satisfies the following ward identity

$$k_\mu \theta_\nu^a T^{\mu\nu} + \theta_\mu^a T^\mu = 0 \quad (4.61)$$

Similar consideration can be applied to the following type II massive scalar gauge state

$$\left[\frac{1}{2} \alpha_{-1} \cdot \alpha_{-1} + \frac{5}{2} k \cdot \alpha_{-2} + \frac{3}{2} (k \cdot \alpha_{-1})^2 \right] |k, l=0, i, j\rangle \quad (4.62)$$

which corresponds to a *massive* $U(N)$ symmetry.

2. Chan-Paton soliton ZNS on $R^{25} \otimes T^1$

We now discuss soliton ZNS on torus compactification of bosonic open string. As is well known, the massless $U(N)$ gauge symmetry will be broken in general after compactification unless N D-branes, in the T-dual picture, are coincident. We will see that when D -branes are coincident, one has enhancement of (unwinding) ZNS and the massless $U(N)$ symmetry will be recovered. These ZNS can be considered as charges or symmetry parameters of $U(N)$ group.

In the discussion of open string compactification, one needs to turn on the Wilson line or nonzero background gauge field in the compact direction. This will effect the momentum in the compact direction, and the Virasoro operators become

$$L_0 = \frac{1}{2} \left(\frac{2\pi l - \theta_j + \theta_i}{2\pi R} \right)^2 + \frac{1}{2} (k^\mu)^2 + \sum_{n=1}^{\infty} (\alpha_{-n}^\mu \alpha_n^\mu + \alpha_{-n}^{25} \alpha_n^{25}), \quad (4.63)$$

$$L_m = \frac{1}{2} \sum_{n=-\infty}^{\infty} \vec{\alpha}_{m-n} \cdot \vec{\alpha}_n. \quad (4.64)$$

Note that in Eq.(4.64), $\alpha_0^{25} \equiv p^{25}$ which also appears in the first term in Eq.(4.63). k is the 25d momentum. θ_i, R are the gauge and space-time moduli respectively and l is the winding number in the compact direction. The spectrums of type I and type II ZNS become

$$M^2 = \left(\frac{2\pi l - \theta_j + \theta_i}{2\pi R} \right)^2 + 2I \quad (4.65)$$

and

$$M^2 = \left(\frac{2\pi l - \theta_j + \theta_i}{2\pi R} \right)^2 + 2(I + 1) \quad (4.66)$$

where $I = \sum_{n=1}^{\infty} (\alpha_{-n}^{\mu} \alpha_n^{\mu} + \alpha_{-n}^{25} \alpha_n^{25})$.

For the massless case $I = l = 0$, one gets N^2 massless solution from equation Eq.(4.65)

$$k_{\mu} \alpha_{-1}^{\mu} |k, l = 0, i, j\rangle \quad (4.67)$$

if all θ_i are equal, or in the T-dual picture when N D -branes are coincident. These N^2 massless ZNS correspond to the charges of massless $U(N)$ gauge symmetry. There is no type II massless solution in Eq.(4.66).

We are now ready to discuss the interesting massive case. For $M^2 = 2$ and general moduli (R, θ_i) ,

1. $I = 1, l = 0$, one gets two ZNS solutions from Eq.(4.65):

$$[(\varepsilon \cdot \alpha_{-1})(k \cdot \alpha_{-1}) + \varepsilon \cdot \alpha_{-2}] |k, l = 0, i, i\rangle, \quad \varepsilon \cdot k = 0 \quad (4.68)$$

and

$$(k \cdot \alpha_{-1} \alpha_{-1}^{25} + \alpha_{-2}^{25}) |k, l = 0, i, i\rangle. \quad (4.69)$$

If all θ_i are equal, the (i, i) is enhanced to (i, j) . Eq.(4.69) implies a massive $U(N)$ symmetry with transformation parameter θ^a . Eq.(4.68) implies a massive Einstein-Yang-Mills-type symmetry with transformation parameter θ_{μ}^a

2. $I = 0, \frac{2\pi l - \theta_j + \theta_i}{2\pi R} = \pm\sqrt{2}$, one gets solution from Eq.(4.65)

$$(k \cdot \alpha_{-1} \pm \sqrt{2} \alpha_{-1}^{25}) |k, l, i, j\rangle. \quad (4.70)$$

Now since $|\theta_i - \theta_j| < 2\pi$, for any given R , there is at most one solution of $(|l|, |\theta_i - \theta_j|)$. One is tempted to consider the case

$$\left(k \cdot \alpha_{-1} \pm \sqrt{2}\alpha_{-1}^{25}\right) \left|k, l = \pm\sqrt{2}R, i, i\right\rangle. \quad (4.71)$$

That means in the moduli $(R = \sqrt{2}n, \theta_i)$ with $n \in Z^+$, one has *soliton* ZNS which imply a *massive* $U(1)^N$ symmetry. If all θ_i are equal, the (i, i) is enhanced to (i, j) . Eq.(4.71) implies a massive $U(N)$ symmetry at the *discrete* values of moduli point $R = \sqrt{2}n$. For example, in the T-dual picture, for $R = \sqrt{2}, l = \pm 2$, and if all D -branes are coincident, we have an enhanced massive $U(N)$ symmetry. This phenomenon is very different from the massless case, where one gets enhanced $U(N)$ symmetry at *any* radius R when N D -branes are coincident.

We would like to point out that similar Einstein-Yang-Mills-type symmetry was discovered before in the closed Heterotic string theory. There, however, one could have more than one Yang-Mills indices on the transformation parameters. For the type II states with $M^2 = 2$ in Eq.(4.66), $I = l = 0$. One gets one more $U(N)$ ZNS

$$\left[\frac{1}{2}\alpha_{-1} \cdot \alpha_{-1} + \frac{1}{2}\alpha_{-1}^{25}\alpha_{-1}^{25} + \frac{5}{2}k \cdot \alpha_{-2} + \frac{3}{2}(k \cdot \alpha_{-1})^2\right] |k, l = 0, i, j\rangle \quad (4.72)$$

if all θ_i are equal.

For the general mass level, choosing $I = 0$ and i, j in Eq.(4.64), we have $l/R = \pm M$. For say $R = \sqrt{2}$ and $l = \pm\sqrt{2}M$, which implies

$$M^2 = 2n^2, \quad n = 0, 1, 2, \dots \quad (4.73)$$

So we have Chan-Paton soliton ZNS at any higher massive level of the spectrum. Similar result was found in Eq.(4.49) for the closed string case.

Part II

Stringy symmetries of hard string scattering amplitudes

As we mentioned in the introduction at the beginning of this review, there are two main key ideas to probe symmetry of string theory. These are high energy limit of string scatterings and the decoupling of ZNS in the OCFQ string spectrum. In the part I of this review, we used only the idea of ZNS to calculate stringy symmetries in various approaches. Although the results we obtained are valid to all energies, only very limited stringy symmetries of low mass level states can be calculated except for $2D$ strings. It will be, for example, very complicated to do calculations for states with low spin at higher mass levels. In the part II of this review, we will combine both crucial two ideas to simplify the calculation.

The high energy Ward identities derived from the decoupling of $26D$ open bosonic string ZNS, which combines the two key ideas of probing stringy symmetry, will be used to explicitly prove [27–31, 44] Gross’s two conjectures. An infinite number of linear relations among high energy, fixed angle string scattering amplitudes of different string states can be derived. Moreover, these linear relations can be used to fix the proportionality constants or ratios among high energy, fixed angle scattering amplitudes of different string states algebraically at each fixed mass level.

The part II of this review is organized as following. Chapter V is one of the main part of this review. We will use three different methods to explicitly prove Gross conjectures [27–31]. These are the decoupling of high energy ZNS, the high energy Virasoro constraints and a saddle-point calculation. In addition, we show that the high energy limit of the discrete ZNS in $2D$ string theory constructed in part I form a high energy w_∞ symmetry. This result strongly suggests that the linear relations obtained from decoupling of ZNS in $2D$ string theory are indeed related to the hidden symmetry also for the $26D$ string theory.

In chapter VI, in addition to analyze ZNS in the helicity basis in the OCFQ string spectrum, we will work out ZNS in the light-cone DDF construction of string spectrum and ZNS in the BRST WSFT[15]. In chapter VII, we discuss hard closed string scatterings [35]. The KLT relation [43] will be extensively used. We also discuss string BCJ relation

[37, 40, 123–125], which is of much interest in the recent development of calculation of field theory scattering amplitudes. In chapter VIII, we calculate four classes of hard superstring scattering amplitudes and derive the ratios among them [33]. In chapter IX, we discuss hard string scattering from D-branes/ O-planes, and closed string decays to open string [41, 46, 126]. Finally in chapter X, we discuss both hard open and closed string scatterings in the compact spaces [54, 55].

V. INFINITE LINEAR RELATIONS AMONG HIGH ENERGY, FIXED ANGLE STRING SCATTERING AMPLITUDES

In this chapter, we will use three different methods to explicitly prove Gross conjectures for $26D$ bosonic open string theory. We will also show that the high energy limit of discrete ZNS of $2D$ string constructed in part I form a high energy w_∞ symmetry. We begin with an example [26–28] to do the calculation.

A. The first example

For our purpose here, there are four ZNS at mass level $M^2 = 4$. The complete list of ZNS were calculated in chapter II, and the corresponding Ward identities were calculated to be [13]

$$k_\mu \theta_{\nu\lambda} \mathcal{T}_\chi^{(\mu\nu\lambda)} + 2\theta_{\mu\nu} \mathcal{T}_\chi^{(\mu\nu)} = 0, \quad (5.1)$$

$$\left(\frac{5}{2}k_\mu k_\nu \theta'_\lambda + \eta_{\mu\nu} \theta'_\lambda\right) \mathcal{T}_\chi^{(\mu\nu\lambda)} + 9k_\mu \theta'_\nu \mathcal{T}_\chi^{(\mu\nu)} + 6\theta'_\mu \mathcal{T}_\chi^\mu = 0, \quad (5.2)$$

$$\left(\frac{1}{2}k_\mu k_\nu \theta_\lambda + 2\eta_{\mu\nu} \theta_\lambda\right) \mathcal{T}_\chi^{(\mu\nu\lambda)} + 9k_\mu \theta_\nu \mathcal{T}_\chi^{[\mu\nu]} - 6\theta_\mu \mathcal{T}_\chi^\mu = 0, \quad (5.3)$$

$$\left(\frac{17}{4}k_\mu k_\nu k_\lambda + \frac{9}{2}\eta_{\mu\nu} k_\lambda\right) \mathcal{T}_\chi^{(\mu\nu\lambda)} + (9\eta_{\mu\nu} + 21k_\mu k_\nu) \mathcal{T}_\chi^{(\mu\nu)} + 25k_\mu \mathcal{T}_\chi^\mu = 0, \quad (5.4)$$

where $\theta_{\mu\nu}$ is transverse and traceless, and θ'_λ and θ_λ are transverse vectors. They are polarizations of ZNS. In each equation, we have chosen, say, $v_2(k_2)$ to be the vertex operators constructed from ZNS and $k_\mu \equiv k_{2\mu}$. Note that Eq.(5.3) is the inter-particle Ward identity corresponding to D_2 vector ZNS in Eq.(1.12) obtained by antisymmetrizing those terms which contain $\alpha^\mu_{-1} \alpha^\nu_{-2}$ in the original type I and type II vector ZNS [6]. We will use 1 and 2 for the incoming particles and 3 and 4 for the scattered particles. In Eq.(5.1) to Eq.(5.4),

1,3 and 4 can be any string states (including ZNS) and we have omitted their tensor indices for the cases of excited string states. For example, one can choose $v_1(k_1)$ to be the vertex operator constructed from another ZNS which generates an inter-particle Ward identity of the third massive level. The resulting Ward-identity of Eq.(5.3) then relates scattering amplitudes of particles at different mass level. \mathcal{T}'_χ s in Eqs (5.1-5.4) are the mass level $M^2 = 4$, χ -th order string-loop amplitudes.

At this point, $\{\mathcal{T}_\chi^{(\mu\nu\lambda)}, \mathcal{T}_\chi^{(\mu\nu)}, \mathcal{T}_\chi^\mu\}$ is identified to be the *amplitude triplet* [6] of the spin-three state. $\mathcal{T}_\chi^{[\mu\nu]}$ is obviously identified to be the scattering amplitude of the antisymmetric spin-two state with the same momenta as $\mathcal{T}_\chi^{(\mu\nu\lambda)}$. Eq.(5.3) thus relates the scattering amplitudes of two different string states at mass level $M^2 = 4$. Note that Eq.(5.1) to Eq.(5.4) are valid order by order and are *automatically* of the identical form in string perturbation theory. This is consistent with Gross's argument through the calculation of high energy scattering amplitudes. However, it is important to note that Eq.(5.1) to Eq.(5.4) are, in contrast to the high energy $\alpha' \rightarrow \infty$ result of Gross, valid to *all* energy α' and their coefficients do depend on the center of mass scattering angle ϕ_{CM} , which is defined to be the angle between \vec{k}_1 and \vec{k}_3 , through the dependence of momentum k .

We will calculate high energy limit of Eq.(5.1) to Eq.(5.4) without referring to the saddle point calculation in [1–5]. Let's define the normalized polarization vectors

$$e_P = \frac{1}{m_2}(E_2, k_2, 0) = \frac{k_2}{m_2}, \quad (5.5)$$

$$e_L = \frac{1}{m_2}(k_2, E_2, 0), \quad (5.6)$$

$$e_T = (0, 0, 1) \quad (5.7)$$

in the CM frame contained in the plane of scattering. They satisfy the completeness relation

$$\eta^{\mu\nu} = \sum_{\alpha, \beta} e_\alpha^\mu e_\beta^\nu \eta^{\alpha\beta} \quad (5.8)$$

where $\mu, \nu = 0, 1, 2$ and $\alpha, \beta = P, L, T$. *Diag* $\eta^{\mu\nu} = (-1, 1, 1)$. One can now transform all μ, ν coordinates in Eq.(5.1) to Eq.(5.4) to coordinates α, β . For Eq.(5.1), we have $\theta^{\mu\nu} = e_L^\mu e_L^\nu - e_T^\mu e_T^\nu$ or $\theta^{\mu\nu} = e_L^\mu e_T^\nu + e_T^\mu e_L^\nu$. In the high energy $E \rightarrow \infty$, fixed angle ϕ_{CM} limit, one identifies $e_P = e_L$ and Eq.(5.1) gives (we drop loop order χ here to simplify the notation)

$$\mathcal{T}_{LLL}^{6\rightarrow 4} - \mathcal{T}_{LTT}^4 + \mathcal{T}_{(LL)}^4 - \mathcal{T}_{(TT)}^2 = 0, \quad (5.9)$$

$$\mathcal{T}_{LLT}^{5\rightarrow 3} + \mathcal{T}_{(LT)}^3 = 0. \quad (5.10)$$

In Eq.(5.9) and Eq.(5.10), we have assigned a relative energy power for each amplitude. For each longitudinal L component, the order is E^2 and for each transverse T component, the order is E . This is due to the definitions of e_L and e_T in Eq.(5.6) and Eq.(5.7), where e_L got one energy power more than e_T . By Eq.(5.9), the E^6 term of the energy expansion for \mathcal{T}_{LLL} is forced to be zero. As a result, the possible leading order term is E^4 . Similar rule applies to \mathcal{T}_{LLT} in Eq.(5.10). For Eq.(5.2), we have $\theta^\mu = e_L^\mu$ or $\theta^\mu = e_T^\mu$ and one gets, in the high energy limit,

$$10\mathcal{T}_{LLL}^{6\rightarrow 4} + \mathcal{T}_{LTT}^4 + 18\mathcal{T}_{(LL)}^4 + 6\mathcal{T}_L^2 = 0, \quad (5.11)$$

$$10\mathcal{T}_{LLT}^{5\rightarrow 3} + \mathcal{T}_{TTT}^3 + 18\mathcal{T}_{(LT)}^3 + 6\mathcal{T}_T^1 = 0. \quad (5.12)$$

For the D_2 Ward identity, Eq.(5.3), we have $\theta^\mu = e_L^\mu$ or $\theta^\mu = e_T^\mu$ and one gets, in the high energy limit,

$$\mathcal{T}_{LLL}^{6\rightarrow 4} + \mathcal{T}_{LTT}^4 + 9\mathcal{T}_{[LL]}^{4\rightarrow 2} - 3\mathcal{T}_L^2 = 0, \quad (5.13)$$

$$\mathcal{T}_{LLT}^{5\rightarrow 3} + \mathcal{T}_{TTT}^3 + 9\mathcal{T}_{[LT]}^3 - 3\mathcal{T}_T^1 = 0. \quad (5.14)$$

It is important to note that $\mathcal{T}_{[LL]}$ in Eq.(5.13) originate from the high energy limit of $\mathcal{T}_{[PL]}$, and the antisymmetric property of the tensor forces the leading E^4 term to be zero. Finally the singlet zero norm state Ward identity, Eq.(5.4), imply, in the high energy limit,

$$34\mathcal{T}_{LLL}^{6\rightarrow 4} + 9\mathcal{T}_{LTT}^4 + 84\mathcal{T}_{(LL)}^4 + 9\mathcal{T}_{(TT)}^2 + 50\mathcal{T}_L^2 = 0. \quad (5.15)$$

One notes that all components of high energy amplitudes of symmetric spin three and antisymmetric spin two states appear at least once in Eq.(5.9) to Eq.(5.15). It is now easy to see that the naive leading order amplitudes corresponding to E^4 appear in Eq.(5.9), (5.11), Eq.(5.13) and Eq.(5.15). However, a simple calculation shows that $\mathcal{T}_{LLL}^4 = \mathcal{T}_{LTT}^4 = \mathcal{T}_{(LL)}^4 = 0$. So the real leading order amplitudes correspond to E^3 , which appear in Eq.(5.10), Eq.(5.12) and Eq.(5.14). A simple calculation shows that [26, 28]

$$\mathcal{T}_{TTT}^3 : \mathcal{T}_{LLT}^3 : \mathcal{T}_{(LT)}^3 : \mathcal{T}_{[LT]}^3 = 8 : 1 : -1 : -1. \quad (5.16)$$

Note that these proportionality constants are, as conjectured by Gross [3, 4], independent of the scattering angle ϕ_{CM} and the loop order χ of string perturbation theory. They are also independent of particles chosen for vertex $v_{1,3,4}$. The ratios in Eq.(5.16) should be *measurable* if the energy scale of string theory is not Planckian. *Most importantly, we now understand that the ratios originate from ZNSs in the OCFQ spectrum of the string!*

The subleading order amplitudes corresponding to E^2 appear in Eq.(5.9), (5.11), Eq.(5.13) and Eq.(5.15). One has 6 unknown amplitudes and 4 equations. Presumably, they are not proportional to each other or the proportional coefficients do depend on the scattering angle ϕ_{CM} . We will justify this point later in our sample calculation. Our calculation here is purely algebraic without any integration and is independent of saddle point calculation in [1–5].

It is important to note that our result in Eq.(5.16) is gauge invariant as it should be since we derive it from Ward identities Eq.(5.1) to Eq.(5.4). On the other hand, the result obtained in [5] with $\mathcal{T}_{TTT}^3 \propto \mathcal{T}_{[LT]}^3$, and $\mathcal{T}_{LLT}^3 = 0$ in the leading order energy at this mass level is, on the contrary, *not* gauge invariant. In fact, with $\mathcal{T}_{LLT}^3 = 0$, an *inconsistency* arises [26–28], for example, between Eq.(5.10) and Eq.(5.12).

We give one example here [26–28] to illustrate the meaning of the massive gauge invariant amplitude. To be more specific, we will use two different gauge choices to calculate the high energy scattering amplitude of symmetric spin three state. The first gauge choice is

$$(\epsilon_{\mu\nu\lambda}\alpha_{-1}^{\mu\nu\lambda} + \epsilon_{(\mu\nu}\alpha_{-1}^{\mu}\alpha_{-2}^{\nu)})|0, k\rangle; \epsilon_{(\mu\nu)} = -\frac{3}{2}k^\lambda\epsilon_{\mu\nu\lambda}, k^\mu k^\nu\epsilon_{\mu\nu\lambda} = 0, \eta^{\mu\nu}\epsilon_{\mu\nu\lambda} = 0. \quad (5.17)$$

In the high energy limit, using the helicity decomposition and writing $\epsilon_{\mu\nu\lambda} = \Sigma_{\mu,\nu,\lambda}e_\mu^\alpha e_\nu^\beta e_\lambda^\delta u_{\alpha\beta\delta}; \alpha, \beta, \delta = P, L, T$, we get

$$\begin{aligned} (\epsilon_{\mu\nu\lambda}\alpha_{-1}^{\mu\nu\lambda} + \epsilon_{(\mu\nu}\alpha_{-1}^{\mu}\alpha_{-2}^{\nu)})|0, k\rangle &= [u_{PLT}(6\alpha_{-1}^{PLT} + 6\alpha_{-1}^{(L}\alpha_{-2}^{T)}) \\ &\quad + u_{TTP}(3\alpha_{-1}^{TTP} - 3\alpha_{-1}^{LLP} + 3\alpha_{-1}^{(T}\alpha_{-2}^{T)} - 3\alpha_{-1}^{(L}\alpha_{-2}^{L)}) \\ &\quad + u_{TTL}(3\alpha_{-1}^{TTL} - \alpha_{-1}^{LLL}) + u_{TTT}(\alpha_{-1}^{TTT} - 3\alpha_{-1}^{LLT})]|0, k\rangle. \end{aligned} \quad (5.18)$$

The second gauge choice is

$$\tilde{\varepsilon}_{\mu\nu\lambda}\alpha_{-1}^{\mu\nu\lambda}|0,k\rangle;k^\mu\tilde{\varepsilon}_{\mu\nu\lambda}=0,\eta^{\mu\nu}\tilde{\varepsilon}_{\mu\nu\lambda}=0. \quad (5.19)$$

In the high energy limit, similar calculation gives

$$\tilde{\varepsilon}_{\mu\nu\lambda}\alpha_{-1}^{\mu\nu\lambda}|0,k\rangle=[\tilde{u}_{TTL}(3\alpha_{-1}^{TTL}-\alpha_{-1}^{LLL})+\tilde{u}_{TTT}(\alpha_{-1}^{TTT}-3\alpha_{-1}^{LLT})]|0,k\rangle. \quad (5.20)$$

It is now easy to see that the first and second terms of Eq.(5.18) will not contribute to the high energy scattering amplitude of the symmetric spin three state due to the spin two Ward identities Eq.(5.10) and Eq.(5.9) if we identify $e_P = e_L$. Thus the two different gauge choices Eq.(5.17) and Eq.(5.19) give the same high energy scattering amplitude. It can be shown that this massive gauge symmetry is valid to all energy and is the result of the decoupling of massive spin two ZNS at mass level $M^2 = 4$. Note that the α_{-1}^{LLT} term of Eq.(5.20), which corresponds to the amplitude \mathcal{T}_{LLT}^3 , was missing in the calculation of Ref [5]. The issue was discussed in details in [29]. To further justify our result, we give a sample calculation in the next section.

1. A sample calculation of mass level $M^2 = 4$

In this section, we give a detailed calculation of a set of sample scattering amplitudes to explicitly justify our results presented in the last section. Since the proportionality constants in Eq.(5.16) are independent of particles chosen for vertex $v_{1,3,4}$, for simplicity, we will choose them to be tachyons. For the string-tree level $\chi = 1$, with one tensor v_2 and three tachyons $v_{1,3,4}$, all scattering amplitudes of mass level $M_2^2 = 4$ were calculated in [13]. They are ($s - t$ channel only)

$$\begin{aligned} \mathcal{T}^{\mu\nu\lambda} &= \int \prod_{i=1}^4 dx_i \langle e^{ik_1 X} \partial X^\mu \partial X^\nu \partial X^\lambda e^{ik_2 X} e^{ik_3 X} e^{ik_4 X} \rangle \\ &= \frac{\Gamma(-\frac{s}{2}-1)\Gamma(-\frac{t}{2}-1)}{\Gamma(\frac{u}{2}+2)} [-t/2(t^2/4-1)k_1^\mu k_1^\nu k_1^\lambda + 3(s/2+1)t/2(t/2+1)k_1^{(\mu} k_1^{\nu} k_3^{\lambda)} \\ &\quad - 3s/2(s/2+1)(t/2+1)k_1^{(\mu} k_3^{\nu} k_3^{\lambda)} + s/2(s^2/4-1)k_3^\mu k_3^\nu k_3^\lambda], \end{aligned} \quad (5.21)$$

$$\begin{aligned}
\mathcal{T}^{(\mu\nu)} &= \int \prod_{i=1}^4 dx_i < e^{ik_1 X} \partial^2 X^{(\mu} \partial X^{\nu)} e^{ik_2 X} e^{ik_3 X} e^{ik_4 X} > \\
&= \frac{\Gamma(-\frac{s}{2}-1)\Gamma(-\frac{t}{2}-1)}{\Gamma(\frac{u}{2}+2)} [t/2(t^2/4-1)k_1^\mu k_1^\nu - (s/2+1)t/2(t/2+1)k_1^{(\mu} k_3^{\nu)} \\
&\quad + s/2(s/2+1)(t/2+1)k_3^{(\mu} k_1^{\nu)} - s/2(s^2/4-1)k_3^\mu k_3^\nu], \tag{5.22}
\end{aligned}$$

$$\begin{aligned}
\mathcal{T}^\mu &= \frac{1}{2} \int \prod_{i=1}^4 dx_i < e^{ik_1 X} \partial^3 X^\mu e^{ik_2 X} e^{ik_3 X} e^{ik_4 X} > \\
&= \frac{\Gamma(-\frac{s}{2}-1)\Gamma(-\frac{t}{2}-1)}{\Gamma(\frac{u}{2}+2)} [s/2(s^2/4-1)k_3^\mu - t/2(t^2/4-1)k_1^\mu], \tag{5.23}
\end{aligned}$$

$$\begin{aligned}
\mathcal{T}^{[\mu\nu]} &= \int \prod_{i=1}^4 dx_i < e^{ik_1 X} \partial^2 X^{[\mu} \partial X^{\nu]} e^{ik_2 X} e^{ik_3 X} e^{ik_4 X} > \\
&= \frac{\Gamma(-\frac{s}{2}-1)\Gamma(-\frac{t}{2}-1)}{\Gamma(\frac{u}{2}+2)} [(\frac{s+t}{2})(s/2+1)(t/2+1)k_3^{[\mu} k_1^{\nu]}] \tag{5.24}
\end{aligned}$$

where $s = -(k_1 + k_2)^2$, $t = -(k_2 + k_3)^2$ and $u = -(k_1 + k_3)^2$ are the Mandelstam variables. In deriving Eq.(5.21) to Eq.(5.24), we have made the $SL(2, R)$ gauge fixing by choosing $x_1 = 0, 0 \leq x_2 \leq 1, x_3 = 1, x_4 = \infty$.

To calculate the high energy expansions ($s, t \rightarrow \infty, \frac{s}{t} = \text{fixed}$) of these scattering amplitudes, one needs the following energy expansion formulas [28]

$$e_P.k_1 = (\frac{-2E^2}{M_2})[1 - (\frac{M_2^2 - 2}{4})\frac{1}{E^2}], \tag{5.25}$$

$$e_L.k_1 = (\frac{-2E^2}{M_2})[1 - (\frac{M_2^2 - 2}{4})\frac{1}{E^2} + (\frac{M_2^2}{4})\frac{1}{E^4} + (\frac{M_2^4 - 2M_2^2}{16})\frac{1}{E^6} + O(\frac{1}{E^8})], \tag{5.26}$$

$$e_T.k_1 = 0, \tag{5.27}$$

$$e_P.k_3 = (\frac{E^2}{M_2}) \left\{ 2\xi^2 + [\frac{M_2^2}{2}\eta^2 + (3\xi^2 - 1)]\frac{1}{E^2} + (2\xi^2 - 1)(\frac{M_2^2 + 2}{4})^2\frac{1}{E^6} + O(\frac{1}{E^8}) \right\}, \tag{5.28}$$

$$\begin{aligned}
e_L.k_3 &= (\frac{E^2}{M_2}) \left\{ 2\xi^2 + [-\frac{M_2^2}{2}\eta^2 + (3\xi^2 - 1)]\frac{1}{E^2} + (\frac{M_2^2}{2}\xi^2)\frac{1}{E^4} \right. \\
&\quad \left. + (\frac{M_2^4 - 4M_2^2\xi^2 + 8\xi^2 - 4}{16})\frac{1}{E^6} + O(\frac{1}{E^8}) \right\}, \tag{5.29}
\end{aligned}$$

$$e_T.k_3 = (-2\xi\eta)E - (\frac{2\xi\eta}{E}) + (\frac{\xi\eta}{E^3}) - (\frac{\xi\eta}{E^5}) + O(\frac{1}{E^7}) \quad (5.30)$$

where $\xi = \sin \frac{\phi_{CM}}{2}$ and $\eta = \cos \frac{\phi_{CM}}{2}$. The high energy expansions of Mandelstam variables are given by

$$s = (E_1 + E_2)^2 = 4E^2, \quad (5.31)$$

$$t = (-4\xi^2)E^2 + (M_2^2 - 6)\xi^2 + \frac{1}{8}(M_2^2 + 2)^2(1 - 2\xi^2)\frac{1}{E^4} + O(\frac{1}{E^6}). \quad (5.32)$$

We can now explicitly calculate all amplitudes in Eq.(5.16). After some algebra, we get

$$\mathcal{T}_{TTT} = -8E^9\mathcal{T}(3)\sin^3\phi_{CM}[1 + \frac{3}{E^2} + \frac{5}{4E^4} - \frac{5}{4E^6} + O(\frac{1}{E^8})], \quad (5.33)$$

$$\begin{aligned} \mathcal{T}_{LLT} = & -E^9\mathcal{T}(3)[\sin^3\phi_{CM} + (6\sin\phi_{CM}\cos^2\phi_{CM})\frac{1}{E^2} \\ & - \sin\phi_{CM}(\frac{11}{2}\sin^2\phi_{CM} - 6)\frac{1}{E^4} + O(\frac{1}{E^6})], \end{aligned} \quad (5.34)$$

$$\begin{aligned} \mathcal{T}_{[LT]} = & E^9\mathcal{T}(3)[\sin^3\phi_{CM} - (2\sin\phi_{CM}\cos^2\phi_{CM})\frac{1}{E^2} \\ & + \sin\phi_{CM}(\frac{3}{2}\sin^2\phi_{CM} - 2)\frac{1}{E^4} + O(\frac{1}{E^6})], \end{aligned} \quad (5.35)$$

$$\begin{aligned} \mathcal{T}_{(LT)} = & E^9\mathcal{T}(3)[\sin^3\phi_{CM} + \sin\phi_{CM}(\frac{3}{2} - 10\cos\phi_{CM} \\ & - \frac{3}{2}\cos^2\phi_{CM})\frac{1}{E^2} - \sin\phi_{CM}(\frac{1}{4} + 10\cos\phi_{CM} + \frac{3}{4}\cos^2\phi_{CM})\frac{1}{E^4} + O(\frac{1}{E^6})] \end{aligned} \quad (5.36)$$

where

$$\mathcal{T}(N) = \sqrt{\pi}(-1)^{N-1}2^{-N}E^{-1-2N}(\sin \frac{\phi_{CM}}{2})^{-3}(\cos \frac{\phi_{CM}}{2})^{5-2N}\exp(-\frac{s\ln s + t\ln t - (s+t)\ln(s+t)}{2}) \quad (5.37)$$

is the high energy limit of $\frac{\Gamma(-\frac{s}{2}-1)\Gamma(-\frac{t}{2}-1)}{\Gamma(\frac{u}{2}+2)}$ with $s + t + u = 2N - 8$, and we have calculated it up to the next leading order in E . We thus have justified Eq.(5.16) with

$\mathcal{T}_{TTT}^3 = -8E^9 \mathcal{T}(3) \sin^3 \phi_{CM}$ and $\mathcal{T}_{LLT}^5 = 0$. We have also checked that $\mathcal{T}_{LLL}^6 = \mathcal{T}_{LLL}^4 = \mathcal{T}_{LTT}^4 = \mathcal{T}_{(LL)}^4 = 0$ as claimed in the previous section.

Note that, unlike the leading E^9 order, the angular dependences of E^7 order are different for each amplitudes. The subleading order amplitudes corresponding to \mathcal{T}^2 (E^8 order) appear in Eq.(5.9), (5.11), Eq.(5.13) and Eq.(5.15). One has 6 unknown amplitudes. An explicit sample calculation gives

$$\mathcal{T}_{LLL}^2 = -4E^8 \sin \phi_{CM} \cos \phi_{CM} \mathcal{T}(3), \quad (5.38)$$

$$\mathcal{T}_{LTT}^2 = -8E^8 \sin^2 \phi_{CM} \cos \phi_{CM} \mathcal{T}(3), \quad (5.39)$$

which show that their angular dependences are indeed different or the proportional coefficients do depend on the scattering angle ϕ_{CM} .

2. Results of mass level $M^2 = 6$

The calculations for $M^2 = 4$ in the previous section can be generalized to $M^2 = 6$ [28]. The calculation was however much more tedious, and to the leading order in energy one ended up with 8 equations and 9 amplitudes. A calculation showed that [28]

$$\begin{aligned} \mathcal{T}_{TTTT}^4 : \mathcal{T}_{TTLL}^4 : \mathcal{T}_{LLLL}^4 : \mathcal{T}_{TTL}^4 : \mathcal{T}_{LLL}^4 : \tilde{\mathcal{T}}_{LT,T}^4 : \tilde{\mathcal{T}}_{LP,P}^4 : \mathcal{T}_{LL}^4 : \tilde{\mathcal{T}}_{LL}^4 = \\ 16 : \frac{4}{3} : \frac{1}{3} : -\frac{4\sqrt{6}}{9} : -\frac{\sqrt{6}}{9} : -\frac{2\sqrt{6}}{3} : 0 : \frac{2}{3} : 0. \end{aligned} \quad (5.40)$$

Note that these proportionality constants are again, as conjectured by Gross, independent of the scattering angle ϕ_{CM} and the loop order χ of string perturbation theory. A sample calculation of scattering amplitudes for mass level $M^2 = 6$ [28] justified the ratios above calculated by solving 8 linear relations derived from the decoupling of high energy ZNS in the GR. There are two 0 amplitudes in Eq.(5.40), which mean they are subleading order amplitudes in energy.

It was remarkable to see that the linear relations obtained by high energy limit of stringy Ward identities or decoupling of ZNS were just good enough to solve all the high energy amplitudes in terms of one amplitude! It was even more remarkable to see that the ratios

obtained by solving these linear relations matched exactly with the sample calculations for the high energy string amplitudes. However, the calculation soon becomes too complicated to manage when one goes to even higher mass levels. In the next section, we will adopt another strategy to generalize the calculations to *arbitrary* mass levels.

B. Decoupling of high energy ZNS at arbitrary mass levels

In the following three sections, we will use three methods to generalize our calculations in the previous section to arbitrary mass levels. In this section we will first use method of decoupling of high energy ZNS. We will focus on 4-point functions in this section, although our discussion can be generalized to higher point correlation functions. Due to Poincare symmetry, a 4-point function is a function of merely two parameters. Viewing a 4-point function as the scattering amplitude of a two-body scattering process, one can choose the two parameters to be E (one half of the center of mass energy for the incoming particles i.e., particles 1 and 2 in Fig. 1, and ϕ (the scattering angle between particles 1 and 3). For convenience we will take the center of mass frame and put the momenta of particles 1 and 2 along the X^1 -direction, with the momenta of particles 3 and 4 on the $X^1 - X^2$ plane. The momenta of the particles are

$$k_1 = (\sqrt{p^2 + M_1^2}, -p, 0), \quad (5.41)$$

$$k_2 = (\sqrt{p^2 + M_2^2}, p, 0), \quad (5.42)$$

$$k_3 = (-\sqrt{q^2 + M_3^2}, -q \cos \phi, -q \sin \phi), \quad (5.43)$$

$$k_4 = (-\sqrt{q^2 + M_4^2}, q \cos \phi, q \sin \phi). \quad (5.44)$$

They satisfy $k_i^2 = -m_i^2$. In the high energy limit, the Mandelstam variables are

$$s \equiv -(k_1 + k_2)^2 = 4E^2 + \mathcal{O}(1/E^2), \quad (5.45)$$

$$t \equiv -(k_2 + k_3)^2 = -4 \left(E^2 - \frac{\sum_{i=1}^4 M_i^2}{4} \right) \sin^2 \frac{\phi}{2} + \mathcal{O}(1/E^2), \quad (5.46)$$

$$u \equiv -(k_1 + k_3)^2 = -4 \left(E^2 - \frac{\sum_{i=1}^4 M_i^2}{4} \right) \cos^2 \frac{\phi}{2} + \mathcal{O}(1/E^2), \quad (5.47)$$

where E is related to p and q as

$$E^2 = p^2 + \frac{M_1^2 + M_2^2}{2} = q^2 + \frac{M_3^2 + M_4^2}{2}. \quad (5.48)$$

The polarization bases for the 4 particles are

$$e^L(1) = \frac{1}{M_1}(p, -\sqrt{p^2 + M_1^2}, 0), e^T(1) = (0, 0, -1), \quad (5.49)$$

$$e^L(2) = \frac{1}{M_2}(p, \sqrt{p^2 + M_2^2}, 0), e^T(2) = (0, 0, 1), \quad (5.50)$$

$$e^L(3) = \frac{1}{M_3}(-q, -\sqrt{q^2 + M_3^2} \cos \phi, -\sqrt{q^2 + M_3^2} \sin \phi), e^T(3) = (0, -\sin \phi, \cos \phi), \quad (5.51)$$

$$e^L(4) = \frac{1}{M_4}(-q, \sqrt{q^2 + M_4^2} \cos \phi, \sqrt{q^2 + M_4^2} \sin \phi), e^T(4) = (0, \sin \phi, -\cos \phi). \quad (5.52)$$

The high energy limit under consideration is

$$\alpha' E^2 \rightarrow \infty, \quad \phi = \text{fixed}. \quad (5.53)$$

Based on the saddle-point approximation of Gross and Mende [1, 2], Gross and Manes [5] computed the high energy limit of 4-point functions in the bosonic open string theory. To explain their result, let us first define our notations and conventions. For a particle of momentum k , we define an orthonormal basis of polarizations $\{e^P, e^L, e^{T_i}\}$. The momentum polarization e^P is proportional to k , the longitudinal polarization e^L is the space-like unit vector whose spatial component is proportional to that of k , and e^{T_i} are the space-like unit-vectors transverse to the spatial momentum. As an example, for k pointing along the X^1 -direction,

$$k = (k^0, k^1, k^2, \dots, k^{25}) = (E, p, 0, \dots, 0), \quad p > 0, \quad (5.54)$$

the basis of polarization is

$$e^P = \frac{1}{M}(\sqrt{p^2 + M^2}, p, 0, 0, \dots, 0), \quad e^L = \frac{1}{M}(p, \sqrt{p^2 + M^2}, 0, 0, \dots, 0), \quad e^{T_i} = (0, 0, \dots, 1, \dots), \quad (5.55)$$

where M is the mass of the particle. In general, e^{T_i} (for $i = 3, \dots, 25$) is just the unit vector in the X^i -direction, and the definitions of e^P, e^L and e^{T_2} will depend on the motion of the particle. For e^{T_2} , which is parallel to the scattering plane, we denote it by e^T (see Fig. 1). The orientations of e^{T_2} for each particle are fixed by the right-hand rule, $\vec{k} \times e^{T_2} = e^{T_3}$, where \vec{k} is the spatial momentum of 4-vector k . We will use the notation $\partial^n X^A \equiv e^A \cdot \partial^n X$ for $A = P, L, T, T_i$.

Each vertex is a polynomial of $\{\partial^n X^A\}$ times the exponential factor $\exp(ik \cdot X)$. Among all possible choices of polarizations for the 4 vertices in a 4-point function, we now argue that only the polarizations L and T need to be considered. The polarization P can be gauged

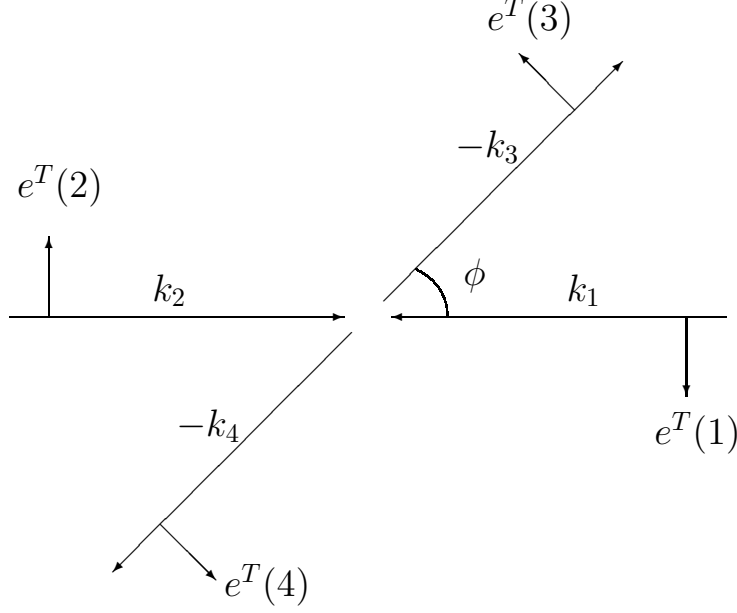


FIG. 1: Kinematic variables in the center of mass frame

away using ZNS [15]. To see why we can ignore all T_i 's except T , we note that a prefactor $\partial^n X^A$ can be contracted with the exponent $ik \cdot X$ of another vertex. The contribution of this contraction to a scattering amplitude is proportional to $k^A \sim E$. If $k^A \neq 0$ (i.e., if $A = L$ or T), this is much more important in the high energy limit than a contraction with another prefactor $\partial^m X^B$, which gives $\eta^{AB} \sim E^0$. Therefore, if all polarizations in all vertices are chosen to be either L or T , the resulting 4-point function will dominate over other choices of polarizations.

The central idea behind the algebraic approach used in [26–28] and [29] was the decoupling of ZNS (i.e., the requirement of stringy gauge invariance). A crucial step in the derivation is to replace the polarization e^P by e^L in the ZNS. It is assumed that, while ZNS decouple at all energies, the replacement leads to states that are decoupled at high energies.

1. Main results

For brevity, we will refer to all 4-point functions different from each other by a single vertex at the same mass level as a “*family*”. When we compare members of a family, we only need to specify the vertex which is changed.

A 4-point function will be said to be *at the leading order* if it is not subleading to any of its *siblings*. We will ignore those that are not at the leading order. Our aim is to find the

numerical ratios of all 4-point functions in the same family at the leading order. Apparently, there are more 4-point functions at the leading order at higher mass levels. Our goal may seem insurmountable at first sight.

Saving the derivation for later, we give our main results here. A 4-point function is at the leading order if and only if the vertex V under comparison is a linear combination of vertices of the form

$$V^{(N,m,q)}(k) = (\partial X^T)^{N-m-2q} (\partial X^L)^{2m} (\partial^2 X^L)^q e^{ik \cdot X}, \quad (5.56)$$

where

$$N \geq 2m + 2q, \quad m, q \geq 0. \quad (5.57)$$

The corresponding states are of the form

$$(\alpha_{-1}^T)^{N-2m-2q} (\alpha_{-1}^L)^{2m} (\alpha_{-2}^L)^q |0, k\rangle. \quad (5.58)$$

The mass squared is $2(N-1)$. All other states involving $\alpha_{-2}^T, \alpha_{-3}^A, \dots$ are subleading.

Using the notation²

$$\mathcal{T}^{(N,m,q)} = \langle V_1 V^{(N,m,q)}(k) V_3 V_4 \rangle, \quad (5.59)$$

all linear relations among different choices of $V^{(N,m,q)}$ (obtained from the decoupling of high energy ZNS) can be solved by the simple expression

$$\lim_{E \rightarrow \infty} \frac{\mathcal{T}^{(N,2m,q)}}{\mathcal{T}^{(N,0,0)}} = \left(-\frac{1}{M}\right)^{2m+q} \left(\frac{1}{2}\right)^{m+q} (2m-1)!!, \quad (5.60)$$

$$\lim_{E \rightarrow \infty} \frac{\mathcal{T}^{(N,2m+1,q)}}{\mathcal{T}^{(N,0,0)}} = 0, \quad (5.61)$$

where $M = \sqrt{2(N-1)}$. This formula tells us how to trade ∂X^L and $\partial^2 X^L$ for ∂X^T , so that all 4-point functions can be related to the one involving only ∂X^T in V_2 . The formula above applies equally well to all vertices.

Since we know the value of a representative 4-point function [26, 28, 29]

$$\mathcal{T}_{N_1 N_2 N_3 N_4}^{T^1 \dots T^2 \dots T^3 \dots T^4 \dots} = (-1)^{N_2 + N_4} [2E^3 \sin \phi_{CM}]^{\Sigma N_i} \mathcal{T}(\Sigma N_i), \quad (5.62)$$

² More rigorously, V_2 needs to be a physical state in order for the correlation function to be well-defined. We should keep in mind that our results should be applied to suitable linear combinations of Eq.(5.58), possibly together with subleading states, to satisfy Virasoro constraints.

where $\mathcal{T}(N)$ is the high-energy limit of $\frac{\Gamma(-\frac{s}{2}-1)\Gamma(-\frac{t}{2}-1)}{\Gamma(\frac{u}{2}+2)}$ with $s + t + u = 2\Sigma N_i - 8$, and we have calculated it up to the next leading order in E in Eq.(5.37), we can immediately write down the explicit expression of a 4-point function if all vertices are nontrivial at the leading order. In Eq.(5.62), N_i is the number of T^i of the i -th vertex operators and T^i is the transverse direction of the i -th particle.

2. Decoupling of high energy ZNS

Before we go on, we recall some terminology used in the old covariant quantization. A state $|\psi\rangle$ in the Hilbert space is *physical* if it satisfies the Virasoro constraints

$$(L_n - \delta_n^0) |\psi\rangle = 0, \quad n \geq 0. \quad (5.63)$$

Since $L_n^\dagger = L_{-n}$, states of the form

$$L_{-n} |\chi\rangle \quad (5.64)$$

are orthogonal to all physical states, and they are called spurious states. ZNS are spurious states that are also physical. They correspond to gauge symmetries. In the old covariant first quantization spectrum of open bosonic string theory, the solutions of physical state conditions include positive-norm propagating states and two types of ZNS. In this section we derive the linear relations among all amplitudes in the same family by taking the high energy limit of ZNS (HZNS). The first step in the derivation is to identify the class of states that are relevant, i.e., those at the leading order. As we explained before, we only need to consider the polarizations e^T and e^L .

To get a rough idea about how each vertex operator scales with E in the high energy limit, we associate a naive dimension to each prefactor $\partial^m X^A$ according to the following rule

$$\partial^m X^T \rightarrow 1, \quad \partial^m X^L \rightarrow 2. \quad (5.65)$$

The reason is the following. Each factor of $\partial^m X^\mu$ has the possibility of contracting with the exponent $ik_i \cdot X$ of another vertex operator so that it scales like E in the high energy limit. Furthermore, components of the polarization vectors e^T and e^L scale with E like E^0 and E^1 , respectively.

When we compare vertex operators at the same mass level, the sum of all the integers m in $\partial^m X^A$ is fixed. Roughly speaking, it is advantageous to have many ∂X^A than having

fewer number of $\partial^m X^A$ with $m > 1$. For example, at the first massive level, the vertex operator $\partial X^T \partial X^T e^{ik \cdot X}$ has a larger naive dimension than $\partial^2 X^T e^{ik \cdot X}$.

The counting of the naive dimension does not take into consideration the possibility that the coefficient of the leading order term happens to vanish by cancellation. The true dimension of a vertex operator can be lower than its naive dimension, although the reverse never happens.

Through experiences accumulated from explicit computations [26–29], we find that the highest spin vertex

$$(\partial X^T)^N e^{ik \cdot X} \leftrightarrow (\alpha_{-1}^T)^N |0, k\rangle \quad (5.66)$$

is always at the leading order in its family. Since the naive dimension of this state equals its true dimension, any state with a lower naive dimension than this vertex operator can be ignored. This implies that we can immediately throw away a lot of vertex operators at each mass level, but there are still many left. The problem is that, although there are disadvantages to have $\partial^m X^T$ with $m \geq 2$ or $\partial^m X^L$ with $m \geq 3$ compared with having $(\partial X^T)^m$, it may be possible that having extra factors of ∂X^L , which has a higher naive dimension than ∂X^T , can compensate the disadvantage of these factors. However, explicit computations at the first few massive levels showed that this never happens.

We will now argue why this is generically true, and show in this section that the only states that will survive the high energy limit at level N are of the form

$$|N, 2m, q\rangle \equiv (\alpha_{-1}^T)^{N-2m-2q} (\alpha_{-1}^L)^{2m} (\alpha_{-2}^L)^q |0; k\rangle. \quad (5.67)$$

Our argument is essentially based on the decoupling of ZNS in the high energy limit. Thanks to the Virasoro algebra, we only need two Virasoro operators

$$L_{-1} = \frac{1}{2} \sum_{n \in \mathbb{Z}} \alpha_{-1+n} \cdot \alpha_{-n} = M \alpha_{-1}^P + \alpha_{-2} \cdot \alpha_{-1} + \cdots, \quad (5.68)$$

$$L_{-2} = \frac{1}{2} \sum_{n \in \mathbb{Z}} \alpha_{-2+n} \cdot \alpha_{-n} = \frac{1}{2} \alpha_{-1} \cdot \alpha_{-1} + M \alpha_{-2}^P + \alpha_{-3} \cdot \alpha_{-1} + \cdots \quad (5.69)$$

to generate all high energy ZNS or HZNS. Here M is the mass operator, i.e., $M^2 = -k^2$ when acting on the state $|0, k\rangle$.

a. Irrelevance of other states To prove that only states of the form Eq.(5.67) are at the leading order, we shall prove that (i) any state which has an odd number of α_{-1}^L is irrelevant

(i.e., subleading in the high energy limit), and (ii) any state involving a creation operator whose naive dimension is less than its mode index n , i.e., states belonging to

$$\{\alpha_{-n}^L, \quad n > 2; \quad \alpha_{-m}^T, \quad m > 1\} \quad (5.70)$$

is also irrelevant. We proceed by mathematical induction.

First we prove that any state which has a single factor of α_{-1}^L is irrelevant, and that any state with two α_{-1}^L 's is irrelevant if it contains an operator of naive dimension less than its index.

Consider the HZNS $L_{-1}\chi$ where χ is any state without any α_{-1}^L , and it is at level $(N-1)$. Note that, except α_{-1}^L , the naive dimension of an operator is always less than or equal to its index (we exclude α_{-1}^P as mentioned above). This means that the naive dimension of χ is less than or equal to $(N-1)$. Since we know that at level N , the state Eq.(5.67) has true dimension N , when computing $L_{-1}\chi$ in the high energy limit, we can ignore everything with naive dimension less than N . This means that we need L_{-1} to increase the naive dimension of χ by no less than 1. In the high energy limit of L_{-1}

$$L_{-1} \rightarrow M\alpha_{-1}^L + \alpha_{-2}^L a_{-1}^L + \alpha_{-2}^T \alpha_{-1}^T + \cdots, \quad (5.71)$$

only the first term will increase the naive dimension of χ by 1. All the rest do not change the naive dimension. This means that, to the leading order,

$$L_{-1}\chi \sim M\alpha_{-1}^L\chi. \quad (5.72)$$

This is a state with a single factor of α_{-1}^L and it is a HZNS, so it should be decoupled in the high energy limit.

Now consider an arbitrary state χ at level $(N-1)$ which has a single factor of α_{-1}^L . If χ involves any operator whose naive dimension is less than its index, the naive dimension of χ is at most $(N-1)$. In the high energy limit

$$L_{-1}\chi \rightarrow M\alpha_{-1}^L\chi + \alpha_{-2}^L\alpha_{-1}^L\chi + \cdots, \quad (5.73)$$

except the first two terms, all other terms are irrelevant because they contain a single factor of α_{-1}^L . As the second term has a naive dimension $(n-1)$ and can be ignored, we conclude that $\alpha_{-1}^L\chi$ is irrelevant.

The next step in mathematical induction is to show that if (a) states with $(2m - 1)$ factors of α_{-1}^L are irrelevant, and (b) states with $2m$ factors of α_{-1}^L are still irrelevant if it also contains any of the operators in Eq.(5.70), then we can prove that both statements are also valid for $m \rightarrow m + 1$.

Suppose χ is an arbitrary state at level $(N - 1)$ which has $2m$ factors of α_{-1}^L 's. The high energy limit of $L_{-1}\chi$ is given by Eq.(5.73). The second term has $(2m - 1)$ factors of α_{-1}^L and is irrelevant. The rest of the terms, except the first, are irrelevant because they contain at least one operator from the set Eq.(5.70). Hence the first term is a HZNS and is irrelevant. We have proved our first claim for $(m + 1)$, i.e., a state with $(2m + 1)$ factors of α_{-1}^L decouple at high energies.

Similarly, consider the case when χ is at level $(N - 1)$ and has $(2m - 1)$ factors of α_{-1}^L . Furthermore we assume that it involves operators from the set Eq.(5.70). Then the first term in Eq.(5.73) is what we want to prove to be irrelevant. The second term is irrelevant because we have just proved that a state with $(2m + 1)$ factors of α_{-1}^L is irrelevant. The rest of the terms are irrelevant because they have $(2m - 1)$ α_{-1}^L 's. Thus we conclude that both claims are correct for $m + 1$ as well. The mathematical induction is complete.

b. Examples of high energy ZNS at low-lying mass levels In this section, we explicitly calculate high energy ZNS (HZNS) of some low-lying mass level. We will also show that the decoupling of these HZNS can be used to derive the desired linear relations. In the old covariant first quantization spectrum of open bosonic string theory, the solutions of physical state conditions include positive-norm propagating states and two types of ZNS. Based on a simplified calculation of higher mass level positive-norm states in [106], some general solutions of ZNS of Eqs.(1.1) and (1.2) at arbitrary mass level were calculated in [94]. Eqs.(1.1) and (1.2) can be derived from Kac determinant in conformal field theory. While type I states have zero-norm at any spacetime, type II states have zero-norm *only* at $D = 26$.

The solutions of Eqs.(1.1) and (1.2) up to the mass level $M^2 = 4$ were calculated in part I of this review. For illustration, let's repeat them and list as follows :

1. $M^2 = -k^2 = 0$:

$$L_{-1} |x\rangle = k \cdot \alpha_{-1} |0, k\rangle ; |x\rangle = |0, k\rangle ; |x\rangle = |0, k\rangle . \quad (5.74)$$

2. $M^2 = -k^2 = 2$:

$$(L_{-2} + \frac{3}{2}L_{-1}^2) |\tilde{x}\rangle = [\frac{1}{2}\alpha_{-1} \cdot \alpha_{-1} + \frac{5}{2}k \cdot \alpha_{-2} + \frac{3}{2}(k \cdot \alpha_{-1})^2] |0, k\rangle; |\tilde{x}\rangle = |0, k\rangle, \quad (5.75)$$

$$L_{-1} |x\rangle = [\theta \cdot \alpha_{-2} + (k \cdot \alpha_{-1})(\theta \cdot \alpha_{-1})] |0, k\rangle; |x\rangle = \theta \cdot \alpha_{-1} |0, k\rangle, \theta \cdot k = 0. \quad (5.76)$$

3. $M^2 = -k^2 = 4$:

$$\begin{aligned} (L_{-2} + \frac{3}{2}L_{-1}^2) |\tilde{x}\rangle &= \{4\theta \cdot \alpha_{-3} + \frac{1}{2}(\alpha_{-1} \cdot \alpha_{-1})(\theta \cdot \alpha_{-1}) + \frac{5}{2}(k \cdot \alpha_{-2})(\theta \cdot \alpha_{-1}) \\ &\quad + \frac{3}{2}(k \cdot \alpha_{-1})^2(\theta \cdot \alpha_{-1}) + 3(k \cdot \alpha_{-1})(\theta \cdot \alpha_{-2})\} |0, k\rangle; \\ |\tilde{x}\rangle &= \theta \cdot \alpha_{-1} |0, k\rangle, k \cdot \theta = 0, \end{aligned} \quad (5.77)$$

$$\begin{aligned} L_{-1} |x\rangle &= [2\theta_{\mu\nu}\alpha_{-1}^\mu\alpha_{-2}^\nu + k_\lambda\theta_{\mu\nu}\alpha_{-1}^\lambda\alpha_{-1}^\mu\alpha_{-1}^\nu] |0, k\rangle; \\ |x\rangle &= \theta_{\mu\nu}\alpha_{-1}^{\mu\nu} |0, k\rangle, k \cdot \theta = \eta^{\mu\nu}\theta_{\mu\nu} = 0, \theta_{\mu\nu} = \theta_{\nu\mu}, \end{aligned} \quad (5.78)$$

$$\begin{aligned} L_{-1} |x\rangle &= [\frac{1}{2}(k \cdot \alpha_{-1})^2(\theta \cdot \alpha_{-1}) + 2\theta \cdot \alpha_{-3} + \frac{3}{2}(k \cdot \alpha_{-1})(\theta \cdot \alpha_{-2}) \\ &\quad + \frac{1}{2}(k \cdot \alpha_{-2})(\theta \cdot \alpha_{-1})] |0, k\rangle; \\ |x\rangle &= [2\theta \cdot \alpha_{-2} + (k \cdot \alpha_{-1})(\theta \cdot \alpha_{-1})] |0, k\rangle, \theta \cdot k = 0, \end{aligned} \quad (5.79)$$

$$\begin{aligned} L_{-1} |x\rangle &= [\frac{17}{4}(k \cdot \alpha_{-1})^3 + \frac{9}{2}(k \cdot \alpha_{-1})(\alpha_{-1} \cdot \alpha_{-1}) + 9(\alpha_{-1} \cdot \alpha_{-2}) \\ &\quad + 21(k \cdot \alpha_{-1})(k \cdot \alpha_{-2}) + 25(k \cdot \alpha_{-3})] |0, k\rangle; \\ |x\rangle &= [\frac{25}{2}k \cdot \alpha_{-2} + \frac{9}{2}\alpha_{-1} \cdot \alpha_{-1} + \frac{17}{4}(k \cdot \alpha_{-1})^2] |0, k\rangle. \end{aligned} \quad (5.80)$$

Note that there are two degenerate vector ZNS, Eq.(5.77) for type II and Eq.(5.79) for type I, at mass level $M^2 = 4$. For mass level $M^2 = 2$, the high energy limit of Eqs. (5.76) and (5.75) are calculated to be

$$L_{-1}(\theta \cdot \alpha_{-1}) |0\rangle \rightarrow \sqrt{2}\alpha_{-1}^L\alpha_{-1}^L + \alpha_{-2}^L |0\rangle; \quad (5.81)$$

$$(L_{-2} + \frac{3}{2}L_{-1}^2) |0\rangle \rightarrow (\sqrt{2}\alpha_{-2}^L + \frac{1}{2}\alpha_{-1}^T\alpha_{-1}^T) |0\rangle \quad (5.82)$$

$$+ \frac{3}{2}(2\alpha_{-1}^L\alpha_{-1}^L + \sqrt{2}\alpha_{-2}^L) |0\rangle. \quad (5.83)$$

Note that Eq.(5.83) is the high energy limit of the second term of type II ZNS. It is easy to see that the decoupling of (5.81) implies the decoupling of (5.83). So one can neglect the

effect of (5.83) even though it is of leading order in energy. It turns out that this phenomena persists to any higher mass level as well. By solving Eqs.(5.81) and (5.82), we get the desired linear relation, $\mathcal{T}_{TT} : \mathcal{T}_L : \mathcal{T}_{LL} = 4 : -\sqrt{2} : 1$. Similarly, the high energy limit of Eqs.(5.77)-(5.80) are calculated to be

$$(L_{-2} + \frac{3}{2}L_{-1}^2) |0\rangle \rightarrow (4\alpha_{-1}^{(T)}\alpha_{-2}^{(L)} + \frac{1}{2}\alpha_{-1}^{(T)}\alpha_{-1}^{(T)}\alpha_{-1}^{(T)}) |0\rangle \quad (5.84)$$

$$+ \frac{3}{2}(4\alpha_{-1}^{(L)}\alpha_{-1}^{(L)}\alpha_{-1}^{(T)} + 4\alpha_{-1}^{(T)}\alpha_{-2}^{(L)}) |0\rangle; \quad (5.85)$$

$$L_{-1}(\theta_{\mu\nu}\alpha_{-1}^{\mu\nu}) |0\rangle \rightarrow [2\alpha_{-1}^{(T)}\alpha_{-2}^{(L)} + 2\alpha_{-1}^{(L)}\alpha_{-1}^{(L)}\alpha_{-1}^{(T)}] |0\rangle; \quad (5.86)$$

$$L_{-1}[2\theta \cdot \alpha_{-2} + (k \cdot \alpha_{-1})(\theta \cdot \alpha_{-1})] |0\rangle \rightarrow (4\alpha_{-1}^{(L)}\alpha_{-1}^{(L)}\alpha_{-1}^{(T)} + 4\alpha_{-1}^{(T)}\alpha_{-2}^{(L)}) |0\rangle; \quad (5.87)$$

$$L_{-1}[\frac{25}{2}k \cdot \alpha_{-2} + \frac{9}{2}\alpha_{-1} \cdot \alpha_{-1} + \frac{17}{4}(k \cdot \alpha_{-1})^2] |0\rangle \rightarrow 0. \quad (5.88)$$

It is easy to see that the decoupling of Eq.(5.86) or (5.87) implies the decoupling of Eq.(5.85). By solving the equations, one gets $\mathcal{T}_{TTT} : \mathcal{T}_{LLT} : \mathcal{T}_{(LT)} : \mathcal{T}_{[LT]} = 8 : 1 : -1 : -1$ which agrees with Eq.(5.16).

c. Linear relations for general mass level According to the previous section, only states of the form (5.67) are relevant in the high energy limit. The mass of the state is $\sqrt{2(N-1)}$. The 4-point function associated with $|N, m, q\rangle$ will be denoted $\mathcal{T}^{(N,m,q)}$. The aim of this section is to find the ratio between a generic $\mathcal{T}^{(N,m,q)}$ and the reference 4-point function, which is taken to be $\mathcal{T}^{(N,0,0)}$.

Consider the type I HZNS calculated from Eq.(1.1)

$$L_{-1}|N-1, 2m-1, q\rangle \simeq M|N, 2m, q\rangle + (2m-1)|N, 2m-2, q+1\rangle, \quad (5.89)$$

where many terms are omitted because they are not of the form (5.67). This implies that

$$\mathcal{T}^{(N,2m,q)} = -\frac{2m-1}{M}\mathcal{T}^{(N,2m-2,q+1)}. \quad (5.90)$$

Using this relation repeatedly, we get

$$\mathcal{T}^{(N,2m,q)} = \frac{(2m-1)!!}{(-M)^m}\mathcal{T}^{(N,0,m+q)}, \quad (5.91)$$

where the double factorial is defined by $(2m-1)!! = \frac{(2m)!}{2^m m!}$.

Next, consider another class of HZNS calculated from type II ZNS in Eq.(1.2)

$$L_{-2}|N-2, 0, q\rangle \simeq \frac{1}{2}|N, 0, q\rangle + M|N, 0, q+1\rangle. \quad (5.92)$$

Again, irrelevant terms are omitted here. From this we deduce that

$$\mathcal{T}^{(N,0,q+1)} = -\frac{1}{2M}\mathcal{T}^{(N,0,q)}, \quad (5.93)$$

which leads to

$$\mathcal{T}^{(N,0,q)} = \frac{1}{(-2M)^q}\mathcal{T}^{(N,0,0)}. \quad (5.94)$$

Our main result Eq.(5.60) is an immediate result of combining Eq.(5.91) and Eq.(5.94).

C. High energy Virasoro constraints

In this section we will establish a “dual description” of our approach explained above. The notion dual to the decoupling of high energy ZNS is Virasoro constraints.

Let us briefly explain how to proceed. First write down a state at a given mass level as linear combination of states of the form Eq.(5.67) with undetermined coefficients, which are interpreted as the Fourier components of spacetime fields. Requiring that the Virasoro generators L_1 and L_2 annihilate the state implies several linear relations on the coefficients. The linear relations can then be solved to obtain ratios among all fields.

To compare the results of the two dual descriptions, we note that the correlation functions can be interpreted as source terms for the particle corresponding to a chosen vertex. Thus the ratios among sources should be the same as the ratios among the fields, since all fields of the same mass have the same propagator. However, some care is needed for the normalization of the field variables. One should use BPZ conjugates to determine the norm of a state and normalize the fields accordingly.

1. Examples

To illustrate how Virasoro constraints can be used to derive linear relations among scattering amplitudes at high energies, we give some explicit examples in this section. We will calculate the proportionality constants among high energy scattering amplitudes of different string states up to mass levels $M^2 = 8$. The results are of course consistent with those of previous work [26–28] using high energy ZNS.

At the mass level $M^2 = 4$, the most general form of physical states at mass level $M^2 = 4$

are given by

$$[\epsilon_{\mu\nu\lambda}\alpha_{-1}^\mu\alpha_{-1}^\nu\alpha_{-1}^\lambda + \epsilon_{(\mu\nu)}\alpha_{-1}^\mu\alpha_{-2}^\nu + \epsilon_{[\mu\nu]}\alpha_{-1}^\mu\alpha_{-2}^\nu + \epsilon_\mu\alpha_{-3}^\mu]|0, k\rangle. \quad (5.95)$$

The Virasoro constraints are

$$\epsilon_{(\mu\nu)} + \frac{3}{2}k^\lambda\epsilon_{\mu\nu\lambda} = 0, \quad (5.96)$$

$$-k^\nu\epsilon_{[\mu\nu]} + 3\epsilon_\mu - \frac{3}{2}k^\nu k^\lambda\epsilon_{\mu\nu\lambda} = 0, \quad (5.97)$$

$$2k^\nu\epsilon_{[\mu\nu]} + 3\epsilon_\mu - 3(k^\nu k^\lambda - \eta^{\nu\lambda})\epsilon_{\mu\nu\lambda} = 0. \quad (5.98)$$

By replacing P by L , and ignoring irrelevant states, one easily gets

$$\epsilon_{TTT} : \epsilon_{(LLT)} : \epsilon_{(LT)} : \epsilon_{[LT]} = 8 : 1 : -3 : -3. \quad (5.99)$$

After including the normalization factor of the field variables³ and the appropriate symmetry factors, one ends up with

$$\begin{aligned} & \mathcal{T}_{TTT} : \mathcal{T}_{(LLT)} : \mathcal{T}_{(LT)} : \mathcal{T}_{[LT]} \\ & = 6\epsilon_{TTT} : 6\epsilon_{(LLT)} : -2\epsilon_{(LT)} : -2\epsilon_{[LT]} = 8 : 1 : -1 : -1, \end{aligned} \quad (5.100)$$

which agrees with Eq.(5.16). It also agree with the results of Eq.(5.60) after Young tableaux decomposition. Here the definitions of \mathcal{T}_{TTT} , $\mathcal{T}_{(LLT)}$, $\mathcal{T}_{(LT)}$, $\mathcal{T}_{[LT]}$ and similar amplitudes hereafter can be found in [26–28] and the result obtained is consistent with the previous ZNS calculation in [26, 27] or Eq.(5.60).

The ratios for $M^2 = 6$ and $M^2 = 8$ can be obtained similarly in Appendix A. At $M^2 = 6$,

$$\begin{aligned} & \mathcal{T}_{TTTT} : \mathcal{T}_{TTLL} : \mathcal{T}_{LLLL} : \mathcal{T}_{TT,L} : \mathcal{T}_{TTL} : \mathcal{T}_{LLL} : \mathcal{T}_{LL} \\ & = 4!\epsilon_{TTTT} : 4!\epsilon_{TTLL} : 4!\epsilon_{LLLL} : -4\epsilon_{TT,L} : -4\epsilon_{TTL} : -4\epsilon_{LLL} : 8\epsilon_{LL}^{(2)} \\ & = 16 : \frac{4}{3} : \frac{1}{3} : -\frac{2\sqrt{6}}{3} : -\frac{4\sqrt{6}}{9} : -\frac{\sqrt{6}}{9} : \frac{2}{3}, \end{aligned} \quad (5.101)$$

which is consistent with the previous ZNS calculation in [28] or Eq.(5.40). It also agree with

³ The normalization factors are determined by the inner product of a state with its BPZ conjugate.

the results of Eq.(5.60) after Young tableaux decomposition. At $M^2 = 8$,

$$\begin{aligned}
& \mathcal{T}_{(TTTTT)} : \mathcal{T}_{(TTTL)} : \mathcal{T}_{(TTLL)} : \mathcal{T}_{(TLLL)} : \mathcal{T}_{(TLLLL)} : \mathcal{T}_{(TLL)} : \mathcal{T}_{T,LL} : \mathcal{T}_{TLL,L} : \mathcal{T}_{TTT,L} \\
&= 5!\epsilon_{(TTTTT)} : 3! \times 2\epsilon_{(TTTL)} : 5!\epsilon_{(TTLL)} : 3! \times 2\epsilon_{(TLLL)} : 5!\epsilon_{(TLLLL)} \\
&: 8\epsilon_{(TLL)} : 8\epsilon_{T,LL} : 3! \times 2\epsilon_{TLL,L} : 3! \times 2\epsilon_{TTT,L} \\
&= 32 : \sqrt{2} : 2 : \frac{3\sqrt{2}}{16} : \frac{3}{8} : \frac{1}{3} : \frac{2}{3} : \frac{\sqrt{2}}{16} : 3\sqrt{2},
\end{aligned} \tag{5.102}$$

which can be checked to be remarkably consistent with the results of Eq.(5.60) after Young tableaux decomposition.

2. General mass levels

In this section we calculate the ratios of string scattering amplitudes in the high energy limit for general mass levels by imposing Virasoro constraints. The final result will, of course, be exactly the same as what we obtained by requiring the decoupling of high energy ZNS. In the presentation here we use the notation of Young's tableaux.

We consider the general mass level $M^2 = 2(N-1)$. The most general state can be written as

$$\begin{aligned}
|N\rangle &= \sum_{\{m_j\}} \left[\left(\frac{1}{1^{m_1} m_1!} \boxed{\mu_1^1 \cdots \mu_{m_1}^1} \alpha_{-1}^{\mu_1^1} \cdots \alpha_{-1}^{\mu_{m_1}^1} \right) \otimes \left(\frac{1}{2^{m_2} m_2!} \boxed{\mu_1^2 \cdots \mu_{m_2}^2} \alpha_{-2}^{\mu_1^2} \cdots \alpha_{-2}^{\mu_{m_2}^2} \right) \otimes \cdots \right] |0, k\rangle \\
&= \sum_{\{m_j\}} \left[\bigotimes_{j=1}^N \left(\frac{1}{j^{m_j} m_j!} \boxed{\mu_1^j \cdots \mu_{m_j}^j} \alpha_{-j}^{\mu_1^j} \cdots \alpha_{-j}^{\mu_{m_j}^j} \right) \right] |0, k\rangle
\end{aligned} \tag{5.103}$$

where $1/(j^{m_j} m_j!)$ are the normalization factors and we defined the abbreviation

$$\alpha_{-j}^{\mu_1^j \cdots \mu_{m_j}^j} \equiv \alpha_{-j}^{\mu_1^j} \cdots \alpha_{-j}^{\mu_{m_j}^j}, \tag{5.104}$$

with m_j is the number of the operator α_{-j} . The summation runs over all possible combinations of m_j 's with the constraints

$$\sum_{j=1}^N j m_j = N \quad \text{and} \quad 0 \leq m_j \leq N, \tag{5.105}$$

so that the total mass is N . Since the upper indices $\{\mu_1^j \cdots \mu_{m_j}^j\}$ in $\alpha_{-j}^{\mu_1^j} \cdots \alpha_{-j}^{\mu_{m_j}^j}$ are symmetric, we used the Young tableaux notation to denote the coefficients in Eq.(5.103). The

direct product \otimes acts on the Young tableaux in the standard way, for example

$$\begin{array}{|c|c|} \hline 1 & 2 \\ \hline \end{array} \otimes \begin{array}{|c|} \hline 3 \\ \hline \end{array} = \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline \end{array} \oplus \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 \\ \hline \end{array} \oplus \begin{array}{|c|c|} \hline 2 & 1 \\ \hline 3 \\ \hline \end{array}. \quad (5.106)$$

To be clear, for example $n = 4$, the state can be written as

$$\begin{aligned} |4\rangle = & \left\{ \frac{1}{4!} \begin{array}{|c|c|c|c|} \hline \mu_1^1 & \mu_2^1 & \mu_3^1 & \mu_4^1 \\ \hline \end{array} \alpha_{-1}^{\mu_1^1} \alpha_{-1}^{\mu_2^1} \alpha_{-1}^{\mu_3^1} \alpha_{-1}^{\mu_4^1} + \frac{1}{2 \cdot 2!} \begin{array}{|c|c|} \hline \mu_1^1 & \mu_2^1 \\ \hline \end{array} \otimes \begin{array}{|c|} \hline \mu_1^2 \\ \hline \end{array} \alpha_{-1}^{\mu_1^1} \alpha_{-1}^{\mu_2^1} \alpha_{-2}^{\mu_1^2} \right. \\ & \left. + \frac{1}{3} \begin{array}{|c|} \hline \mu_1^1 \\ \hline \end{array} \otimes \begin{array}{|c|c|} \hline \mu_1^3 & \mu_2^3 \\ \hline \end{array} \alpha_{-1}^{\mu_1^1} \alpha_{-3}^{\mu_1^3} + \frac{1}{2^2 \cdot 2!} \begin{array}{|c|c|} \hline \mu_1^2 & \mu_2^2 \\ \hline \end{array} \alpha_{-2}^{\mu_1^2} \alpha_{-2}^{\mu_2^2} + \frac{1}{4} \begin{array}{|c|} \hline \mu_1^4 \\ \hline \end{array} \alpha_{-4}^{\mu_1^4} \right\} |0, k\rangle. \quad (5.107) \end{aligned}$$

Next, we will apply the Virasoro constraints to the state Eq.(5.103). The only Virasoro constraints which need to be considered are

$$L_1 |N\rangle = L_2 |N\rangle = 0, \quad (5.108)$$

with L_m the standard Virasoro operator

$$L_m = \frac{1}{2} \sum_{n=-\infty}^{\infty} \alpha_{m+n} \cdot \alpha_{-n}. \quad (5.109)$$

After taking care the symmetries of the Young tableaux, the Virasoro constraints become

$$\begin{aligned} L_1 |N\rangle = & \sum_{\{m_j\}} \left[k^{\mu_1^1} \bigotimes_{j=1}^N \begin{array}{|c|c|c|} \hline \mu_1^j & \dots & \mu_{m_j}^j \\ \hline \end{array} \right. \\ & + \sum_{i=2}^{m_1} \begin{array}{|c|c|c|c|} \hline \mu_2^1 & \dots & \mu_i^1 & \dots & \mu_{m_1}^1 \\ \hline \end{array} \otimes \begin{array}{|c|c|c|} \hline \mu_i^1 & \mu_1^2 & \dots & \mu_{m_2}^2 \\ \hline \end{array} \bigotimes_{j \neq 1,2}^N \begin{array}{|c|c|c|} \hline \mu_1^j & \dots & \mu_{m_j}^j \\ \hline \end{array} \\ & + \sum_{l=3}^N (l-1) \begin{array}{|c|c|c|} \hline \mu_2^1 & \dots & \mu_{m_1}^1 \\ \hline \end{array} \otimes \sum_{i=1}^{m_{l-1}} \begin{array}{|c|c|c|c|} \hline \mu_1^{l-1} & \dots & \mu_i^{l-1} & \dots & \mu_{m_{l-1}}^{l-1} \\ \hline \end{array} \\ & \left. \otimes \begin{array}{|c|c|c|} \hline \mu_i^{l-1} & \mu_1^l & \dots & \mu_{m_l}^l \\ \hline \end{array} \bigotimes_{j \neq 1,l,l-1}^N \begin{array}{|c|c|c|} \hline \mu_1^j & \dots & \mu_{m_j}^j \\ \hline \end{array} \right] \\ & \frac{1}{(m_1-1)!} \alpha_{-1}^{\mu_2^1 \dots \mu_{m_1}^1} \prod_{j \neq 1}^N \left(\frac{1}{j^{m_j} m_j!} \alpha_{-j}^{\mu_1^j \dots \mu_{m_j}^j} \right) |0, k\rangle = 0, \quad (5.110a) \end{aligned}$$

and

$$\begin{aligned}
L_2 |N\rangle = & \sum_{\{m_j\}} \left[\frac{1}{2} \eta^{\mu_1^1 \mu_2^1} \bigotimes_{j=1}^N \boxed{\mu_1^j \cdots \mu_{m_j}^j} \right. \\
& + \boxed{\mu_3^1 \cdots \mu_{m_1}^1} \otimes \boxed{\mu_1^2 \cdots \mu_{m_2+1}^2} k^{\mu_{m_2+1}^2} \bigotimes_{j \neq 1,2}^N \boxed{\mu_1^j \cdots \mu_{m_j}^j} \\
& + \sum_{i=3}^{m_1} \boxed{\mu_3^1 \cdots \hat{\mu}_i^1 \cdots \mu_{m_1}^1} \otimes \boxed{\mu_i^1 \mu_1^3 \cdots \mu_{m_3}^3} \bigotimes_{j \neq 1,3}^N \boxed{\mu_1^j \cdots \mu_{m_j}^j} \\
& + \sum_{l=4}^N (l-2) \boxed{\mu_3^1 \cdots \mu_{m_1}^1} \otimes \sum_{i=1}^{m_{l-2}} \boxed{\mu_1^{l-2} \cdots \hat{\mu}_i^{l-2} \cdots \mu_{m_l}^{l-2}} \\
& \left. \otimes \boxed{\mu_i^{l-2} \mu_1^l \cdots \mu_{m_l}^l} \bigotimes_{j \neq 1, l, l-2}^N \boxed{\mu_1^j \cdots \mu_{m_j}^j} \right] \\
& \frac{1}{(m_1-2)!} \alpha_{-1}^{\mu_3^1 \cdots \mu_{m_1}^1} \prod_{j \neq 1}^N \left(\frac{1}{j^{m_j} m_j!} \alpha_{-j}^{\mu_1^j \cdots \mu_{m_j}^j} \right) |0, k\rangle = 0. \tag{5.110b}
\end{aligned}$$

A hat on an index means that the index is skipped there (and it should appear somewhere else). In the above derivation we have used the identity for the Young tableaux

$$\begin{aligned}
\boxed{1 \cdots p} &= \frac{1}{p} [1 + \sigma_{(21)} + \sigma_{(321)} + \cdots \sigma_{(p \cdots 1)}] \boxed{2 \cdots p} \otimes \boxed{1} \\
&= \frac{1}{p} \sum_{i=1}^p \sigma_{(i1)} \boxed{2 \cdots p} \otimes \boxed{1}, \tag{5.111}
\end{aligned}$$

where $\sigma_{(i \cdots j)}$ are permutation operators.

a. High energy limit of Virasoro constraints States which satisfy the Virasoro constraints are physical states. What we are going to show in the following is that, in the high energy limit, the Virasoro constraints turn out to be strong enough to give the linear relationship among the physical states. To take the high energy limit for the Virasoro constraints, we replace the indices (μ_i, ν_i) by L or T with

$$k^{\mu_i} \rightarrow M e^L, \quad \eta^{\mu_1 \mu_2} \rightarrow e^T e^T, \tag{5.112}$$

where M is the mass operator.

The Virasoro constraints (5.110a) and (5.110b) at high energies become (see Appendix B for detail)

$$\underbrace{\boxed{T \cdots T}}_{n-2q-2-2m} \underbrace{\boxed{L \cdots L}}_{2m+2} \otimes \underbrace{\boxed{L \cdots L}}_q = -\frac{2m+1}{M} \underbrace{\boxed{T \cdots T}}_{n-2q-2-2m} \underbrace{\boxed{L \cdots L}}_{2m} \otimes \underbrace{\boxed{L \cdots L}}_{q+1}, \tag{5.113a}$$

$$\underbrace{\boxed{T \cdots T}}_{n-2q-2-2m} \underbrace{\boxed{L \cdots L}}_{2m} \otimes \underbrace{\boxed{L \cdots L}}_{q+1} = -\frac{1}{2M} \underbrace{\boxed{T \cdots T}}_{n-2q-2m} \underbrace{\boxed{L \cdots L}}_{2m} \otimes \underbrace{\boxed{L \cdots L}}_q, \tag{5.113b}$$

where we have renamed $m_2 \rightarrow q$ and $m_1 \rightarrow N - 2q$. By mathematical recursion, Eq.(5.113a) and Eq.(5.113b) lead to

$$\underbrace{\begin{array}{|c|c|c|c|c|} \hline T & \cdots & T & L & \cdots & L \\ \hline \end{array}}_{N-2q-2m} \otimes \underbrace{\begin{array}{|c|c|c|} \hline L & \cdots & L \\ \hline \end{array}}_q = \frac{(2m-1)!! (-M)^q}{(2m+2q-1)!!} \underbrace{\begin{array}{|c|c|c|c|c|} \hline T & \cdots & T & L & \cdots & L \\ \hline \end{array}}_{N-2q-2m} \underbrace{\begin{array}{|c|c|c|} \hline L & \cdots & L \\ \hline \end{array}}_{2m+2q}, \quad (5.114a)$$

$$\underbrace{\begin{array}{|c|c|c|c|c|} \hline T & \cdots & T & L & \cdots & L \\ \hline \end{array}}_{N-2q-2m} \otimes \underbrace{\begin{array}{|c|c|c|} \hline L & \cdots & L \\ \hline \end{array}}_q = \left(-\frac{1}{2M}\right)^q \underbrace{\begin{array}{|c|c|c|c|c|} \hline T & \cdots & T & L & \cdots & L \\ \hline \end{array}}_{N-2m} \underbrace{\begin{array}{|c|c|c|} \hline L & \cdots & L \\ \hline \end{array}}_{2m}. \quad (5.114b)$$

Combining equations (5.114a) and (5.114b), we get

$$\underbrace{\begin{array}{|c|c|c|c|c|} \hline T & \cdots & T & L & \cdots & L \\ \hline \end{array}}_{N-2q-2m} \otimes \underbrace{\begin{array}{|c|c|c|} \hline L & \cdots & L \\ \hline \end{array}}_q = \left(-\frac{1}{2M}\right)^q \frac{(2k-1)!!}{4^m (N-1)^m} \underbrace{\begin{array}{|c|c|c|} \hline T & \cdots & T \\ \hline \end{array}}_N, \quad (5.115)$$

which is equivalent to Eq.(5.60).

To get the ratio for the specific physical states, we make the Young tableaux decomposition

$$\underbrace{\begin{array}{|c|c|c|c|c|} \hline T & \cdots & T & L & \cdots & L \\ \hline \end{array}}_{N-2q-2m} \otimes \underbrace{\begin{array}{|c|c|c|} \hline L & \cdots & L \\ \hline \end{array}}_q = \sum_{l=0}^q \begin{array}{|c|c|c|c|c|} \hline T & \cdots & T & L & \cdots & L \\ \hline L & \cdots & L & & & \\ \hline \end{array} \cdot (l! C_q^l C_{N-2q-2m}^l), \quad (5.116)$$

where $C_q^l = \frac{q!}{l!(q-l)!}$ and we have $(N-2q-2m)$ T 's and $(2m+q-l)$ L 's in the first column, (l) L 's in the second column in the second line of the above equation. Therefore, we obtain

$$\begin{array}{|c|c|c|c|c|} \hline T & \cdots & T & L & \cdots & L \\ \hline L & \cdots & L & & & \\ \hline \end{array} = \frac{1}{\sum_{l=0}^q l! C_q^l C_{N-2q-2m}^l} \left(-\frac{1}{2M}\right)^q \frac{(2m-1)!!}{4^m (N-1)^m} \underbrace{\begin{array}{|c|c|c|} \hline T & \cdots & T \\ \hline \end{array}}_N,$$

which is consistent with the ratios Eqs.(5.16), (5.40), and (5.102) for $M^2 = 4, 6, 8$ respectively.

D. Saddle-point calculation

In previous sections, we have identified the leading high energy amplitudes and derived the ratios among high energy amplitudes for members of a family at given mass levels, based on decoupling principle. While deductive arguments help to clarify the underlying assumptions and solidify the validity of decoupling principle, it is instructive to compare it with a different approach, such as the saddle-point approximation [29]. Therefore, we

shall perform direct calculations to check the results obtained above and make comparisons between these two approaches.

In this section, we give a direct verification of the ratios among leading high energy amplitudes based on the saddle-point method. The four-point amplitudes to be calculated consist of one massive tensor and three tachyons. Since we have shown that in the high energy limit the only relevant states are those corresponding to

$$(\alpha_{-1}^T)^{N-2m-2q}(\alpha_{-1}^L)^{2m}(\alpha_{-2}^L)^q|0, k\rangle, \quad -k^2 = 2(N-1), \quad (5.117)$$

we only need to calculate the following four-point amplitude

$$\mathcal{T}^{(N,2m,q)} \equiv \int \prod_{i=1}^4 dx_i \langle V_1 V_2^{(N,2m,q)} V_3 V_4 \rangle, \quad (5.118)$$

where

$$V_2^{(N,2m,q)} \equiv (\partial X^T)^{N-2m-2q} (\partial X^P)^{2m} (\partial^2 X^P)^q e^{ik_2 X_2}, \quad (5.119)$$

$$V_i \equiv e^{ik_i X_i}, \quad i = 1, 3, 4. \quad (5.120)$$

Notice that here for leading high energy amplitudes we replace the polarization L by P .

Using either path-integral or operator formalism, after $SL(2, R)$ gauge fixing, we obtain the $s - t$ channel contribution to the stringy amplitude at tree level

$$\begin{aligned} \mathcal{T}^{(N,2m,q)} \Rightarrow \int_0^1 dx x^{(1,2)} (1-x)^{(2,3)} & \left[\frac{e^T \cdot k_1}{x} - \frac{e^T \cdot k_3}{1-x} \right]^{N-2m-2q} \\ & \cdot \left[\frac{e^P \cdot k_1}{x} - \frac{e^P \cdot k_3}{1-x} \right]^{2m} \left[-\frac{e^P \cdot k_1}{x^2} - \frac{e^P \cdot k_3}{(1-x)^2} \right]^q. \end{aligned} \quad (5.121)$$

In order to apply the saddle-point method, we need to rewrite the amplitude above into the “canonical form”. That is,

$$\mathcal{T}^{(N,2m,q)}(K) = \int_0^1 dx u(x) e^{-K f(x)}, \quad (5.122)$$

where

$$K \equiv -(1, 2) \rightarrow \frac{s}{2} \rightarrow 2E^2, \quad (5.123)$$

$$\tau \equiv -\frac{(2, 3)}{(1, 2)} \rightarrow -\frac{t}{s} \rightarrow \sin^2 \frac{\phi}{2}, \quad (5.124)$$

$$f(x) \equiv \ln x - \tau \ln(1-x), \quad (5.125)$$

$$u(x) \equiv \left[\frac{(1, 2)}{M} \right]^{2m+q} (1-x)^{-N+2m+2q} (f')^{2m} (f'')^q (-e^T \cdot k_3)^{N-2m-2q}. \quad (5.126)$$

The saddle-point for the integration of moduli, $x = x_0$, is defined by

$$f'(x_0) = 0, \quad (5.127)$$

and we have

$$x_0 = \frac{1}{1-\tau}, \quad 1-x_0 = -\frac{\tau}{1-\tau}, \quad f''(x_0) = (1-\tau)^3 \tau^{-1}. \quad (5.128)$$

From the definition of $u(x)$, it is easy to see that

$$u(x_0) = u'(x_0) = \dots = u^{(2m-1)}(x_0) = 0, \quad (5.129)$$

and

$$u^{(2m)}(x_0) = \left[\frac{(1, 2)}{M} \right]^{2m+q} (1-x_0)^{-N+2m+2q} (2m)! (f_0'')^{2m+q} (-e^T \cdot k_3)^{N-2m-2q}. \quad (5.130)$$

With these inputs, one can easily evaluate the Gaussian integral associated with the four-point amplitudes, Eq.(5.122),

$$\begin{aligned} & \int_0^1 dx \, u(x) e^{-Kf(x)} \\ &= \sqrt{\frac{2\pi}{Kf_0''}} e^{-Kf_0} \left[\frac{u_0^{(2m)}}{2^m m! (f_0'')^m K^m} + O\left(\frac{1}{K^{m+1}}\right) \right] \\ &= \sqrt{\frac{2\pi}{Kf_0''}} e^{-Kf_0} \left[(-1)^{N-q} \frac{2^{N-2m-q} (2m)!}{m! M^{2m+q}} \tau^{-\frac{N}{2}} (1-\tau)^{\frac{3N}{2}} E^N + O(E^{N-2}) \right]. \end{aligned} \quad (5.131)$$

This result shows explicitly that with one tensor and three tachyons, the energy and angle dependence for the high energy four-point amplitudes only depend on the level N , and we can solve for the ratios among high energy amplitudes within the same family,

$$\lim_{E \rightarrow \infty} \frac{\mathcal{T}^{(N, 2m, q)}}{\mathcal{T}^{(N, 0, 0)}} = \frac{(-1)^q (2m)!}{m! (2M)^{2m+q}} = \left(-\frac{2m-1}{M} \right) \dots \left(-\frac{3}{M} \right) \left(-\frac{1}{M} \right) \left(-\frac{1}{2M} \right)^{m+q}, \quad (5.132)$$

which is consistent with Eq.(5.60).

We conclude this section with three remarks. Firstly, from the saddle-point approach, it is easy to see why the product of α_{-1}^P oscillators induce energy suppression. Their contribution to the stringy amplitude is proportional to powers of $f'(x_0)$, which is zero in the leading order calculation. Secondly, one can also understand why only even numbers of α_{-1}^P oscillators will survive for high energy amplitudes based on the structure of Gaussian integral in Eq.(5.122). While for a vertex operator containing $(2m+1)$ α_{-1}^P 's, we

have $u(x_0) = u'(x_0) = \dots = u^{(2m)}(x_0) = 0$, and the leading contribution comes from $u^{(2m+1)}(x_0)(x - x_0)^{2m+1}$, this gives zero since the odd-power moments of Gaussian integral vanish. Finally, for the alert readers, since we only discuss the $s - t$ channel contribution to the scattering amplitudes, the integration range for the x variable seems to devoid of a direct application of saddle-point method. We will discuss this issue in chapter VII.

E. 2D string at high energies

Although we have shown that there exist infinitely many linear relations among 4-point functions which uniquely fix their ratios in the high-energy limit, it is not totally clear that there is a hidden symmetry responsible for it. However, we would like to claim that these linear relations are indeed the manifestation of the long-sought hidden symmetry of string theory, and that we are on the right track of understanding the symmetry. To persuade the readers, we test our claim on a toy model of string theory –the 2D string theory.

While the hidden symmetry of the 26D bosonic string theory is still at large, the w_∞ symmetry of the 2D string theory is much better understood. It is known to be associated with the discrete Polyakov states discussed in chapter III of part I of this review. Let us now check whether the w_∞ symmetry is generated by the high energy limit of ZNS. In [22], explicit expression for a class of discrete ZNS with Polyakov momenta was given in Eq.(3.26). For illustration, let's repeat it here

$$\begin{aligned}
G_{J,M}^+ &= (J + M + 1)^{-1} \int \frac{dz}{2\pi i} [\psi_{1,-1}^+(z) \psi_{J,M+1}^+(0) + \psi_{J,M+1}^+(z) \psi_{1,-1}^+(0)] \\
&\sim (J - M)! \Delta(J, M, -i\sqrt{2}X) \text{Exp} \left[\sqrt{2}(iMX + (J - 1)\phi) \right] \\
&\quad + (-1)^{2J} \sum_{j=1}^{J-M} (J - M - 1)! \int \frac{dz}{2\pi i} \mathcal{D}(J, M, -i\sqrt{2}X(z), j) \\
&\quad \cdot \text{Exp} \left[\sqrt{2}(i(M + 1)X(z) + (J - 1)\phi(z) - X(0)) \right].
\end{aligned} \tag{5.133}$$

Here $\Delta(J, M, -i\sqrt{2}X)$ is defined by

$$\Delta(J, M, -i\sqrt{2}X) = \begin{vmatrix} S_{2J-1} & S_{2J-2} & \cdots & S_{J+M} \\ S_{2J-2} & S_{2J-3} & \cdots & S_{J+M-1} \\ \cdots & \cdots & \cdots & \cdots \\ S_{J+M} & S_{J+M-1} & \cdots & S_{2M+1} \end{vmatrix}, \tag{5.134}$$

where

$$S_k = S_k \left(\left\{ \frac{-i\sqrt{2}}{k!} \partial^k X(0) \right\} \right), \quad \text{and} \quad S_k = 0 \quad \text{if} \quad k < 0, \quad (5.135)$$

and $S_k(\{a_i\})$'s denote the Schur polynomial defined by

$$\exp \left(\sum_{k=1}^{\infty} a_k x^k \right) = \sum_{k=0}^{\infty} S_k(\{a_i\}) x^k. \quad (5.136)$$

$\mathcal{D}(J, M, -i\sqrt{2}X(z), j)$ is defined by a similar expression as Eq.(5.134), but with the j -th row replaced by $\{(-z)^j - 1 - 2J, (-z)^j - 2J, \dots, (-z)^{j-J-M-2}\}$. It was shown [22] that ZNS in Eq.(5.133) generate a w_∞ algebra.

In the high energy limit, the factors $\partial^k X^A$ are generically proportional to a linear combination of the momenta of other vertices, so it scales with energy E . Thus $\mathcal{D}(J, M, -i\sqrt{2}X, j)$ is subleading to $\Delta(J, M, -i\sqrt{2}X)$. Ignoring the second term in Eq.(5.133) for this reason, we see that these ZNS indeed approach to the discrete states ψ_{JM}^+ in Eq.(3.20)! Thus, the w_∞ algebra generated by Eq.(5.133) is identified to w_∞ symmetry in Eq.(3.25). This result strongly suggests that the linear relations among correlation functions obtained from ZNS are indeed related to the hidden symmetry also for the $26D$ strings.

In chapter XV of part III of this review, we will address a similar issue in the RR of $26D$ string theory, where high energy spacetime symmetry is shown to be related to $SL(5, C)$ of the Appell function F_1 . Although we still do not know what is the exact symmetry group of $26D$ strings, or how it acts on states, these works shed new light on the road to finding the answers.

VI. ZNS IN DDF CONSTRUCTION AND WSFT

In this chapter, in addition to the OCFQ scheme, we will identify and calculate [15] the counterparts of ZNS in two other quantization schemes of $26D$ open bosonic string theory, namely, the light-cone DDF [127–129] ZNS and the off-shell BRST ZNS (with ghost) in WSFT. In particular, special attentions are paid to the inter-particle ZNS in all quantization schemes. For the case of off-shell BRST ZNS, we impose the no ghost conditions and exactly recover two types of on-shell ZNS in the OCFQ string spectrum for the first few low-lying mass levels. We then show that off-shell gauge transformations of WSFT are identical to the on-shell stringy gauge symmetries generated by two types of ZNS in the OCFQ string theory.

Our calculations in this chapter serve as the first step to study stringy symmetries in light-cone DDF and BRST string theories, and to bridge the links between different quantization schemes for both on-shell and off-shell string theories. In section A, we first review the calculations of ZNS in OCFQ spectrum. The most general spectrum analysis in the helicity basis, including ZNS, is then given to discuss the inter-particle D_2 ZNS [6, 8, 11, 12] at mass level $M^2 = 4$. We will see that one can use polarization of either one of the two positive-norm states to represent the polarization of the inter-particle ZNS.

In section B, we calculate both type I and type II ZNS in the light-cone DDF string up to mass level $M^2 = 4$. In section C, we first calculate off-shell ZNS with ghosts from linearized gauge transformation of WSFT. After imposing the no ghost conditions on these ZNS, we can exactly reproduce two types of ZNS in OCFQ spectrum for the first few low-lying mass levels. We then show that off-shell gauge transformations of WSFT are identical to the on-shell stringy gauge symmetries generated by two types of ZNS in the generalized massive σ -model approach [6, 8] of string theory.

Based on the ZNS calculations [26–29], we thus have related gauge symmetry of WSFT [16] to the high energy stringy symmetry conjectured by Gross [1–5].

A. ZNS in the OCFQ spectrum

1. ZNS with constraints

In the OCFQ spectrum of open bosonic string theory, the solutions of physical states conditions include positive-norm propagating states and two types of ZNSs. The solutions of ZNS up to the mass level $M^2 = 4$ were calculated in chapter II. We re-list them in the following :

1. $M^2 = -k^2 = 0$:

$$L_{-1} |x\rangle = k \cdot \alpha_{-1} |0, k\rangle ; |x\rangle = |0, k\rangle ; |x\rangle = |0, k\rangle . \quad (6.1)$$

2. $M^2 = -k^2 = 2$:

$$\left(L_{-2} + \frac{3}{2} L_{-1}^2 \right) |\tilde{x}\rangle = \left[\frac{1}{2} \alpha_{-1} \cdot \alpha_{-1} + \frac{5}{2} k \cdot \alpha_{-2} + \frac{3}{2} (k \cdot \alpha_{-1})^2 \right] |0, k\rangle ; |\tilde{x}\rangle = |0, k\rangle , \quad (6.2a)$$

$$L_{-1} |x\rangle = [\theta \cdot \alpha_{-2} + (k \cdot \alpha_{-1})(\theta \cdot \alpha_{-1})] |0, k\rangle; |x\rangle = \theta \cdot \alpha_{-1} |0, k\rangle, \theta \cdot k = 0. \quad (6.2b)$$

3. $M^2 = -k^2 = 4$:

$$\begin{aligned} \left(L_{-2} + \frac{3}{2} L_{-1}^2 \right) |\tilde{x}\rangle &= \left[4\theta \cdot \alpha_{-3} + \frac{1}{2}(\alpha_{-1} \cdot \alpha_{-1})(\theta \cdot \alpha_{-1}) + \frac{5}{2}(k \cdot \alpha_{-2})(\theta \cdot \alpha_{-1}) \right. \\ &\quad \left. + \frac{3}{2}(k \cdot \alpha_{-1})^2(\theta \cdot \alpha_{-1}) + 3(k \cdot \alpha_{-1})(\theta \cdot \alpha_{-2}) \right] |0, k\rangle; \\ |\tilde{x}\rangle &= \theta \cdot \alpha_{-1} |0, k\rangle, k \cdot \theta = 0, \end{aligned} \quad (6.3a)$$

$$\begin{aligned} L_{-1} |x\rangle &= [2\theta_{\mu\nu} \alpha_{-1}^\mu \alpha_{-2}^\nu + k_\lambda \theta_{\mu\nu} \alpha_{-1}^\lambda \alpha_{-1}^\mu \alpha_{-1}^\nu] |0, k\rangle; \\ |x\rangle &= \theta_{\mu\nu} \alpha_{-1}^{\mu\nu} |0, k\rangle, k \cdot \theta = \eta^{\mu\nu} \theta_{\mu\nu} = 0, \theta_{\mu\nu} = \theta_{\nu\mu}, \end{aligned} \quad (6.3b)$$

$$\begin{aligned} L_{-1} |x\rangle &= \left[\frac{1}{2}(k \cdot \alpha_{-1})^2(\theta \cdot \alpha_{-1}) + 2\theta \cdot \alpha_{-3} + \frac{3}{2}(k \cdot \alpha_{-1})(\theta \cdot \alpha_{-2}) \right. \\ &\quad \left. + \frac{1}{2}(k \cdot \alpha_{-2})(\theta \cdot \alpha_{-1}) \right] |0, k\rangle; \\ |x\rangle &= [2\theta \cdot \alpha_{-2} + (k \cdot \alpha_{-1})(\theta \cdot \alpha_{-1})] |0, k\rangle, \theta \cdot k = 0, \end{aligned} \quad (6.3c)$$

$$\begin{aligned} L_{-1} |x\rangle &= \left[\frac{17}{4}(k \cdot \alpha_{-1})^3 + \frac{9}{2}(k \cdot \alpha_{-1})(\alpha_{-1} \cdot \alpha_{-1}) + 9(\alpha_{-1} \cdot \alpha_{-2}) \right. \\ &\quad \left. + 21(k \cdot \alpha_{-1})(k \cdot \alpha_{-2}) + 25(k \cdot \alpha_{-3}) \right] |0, k\rangle; \\ |x\rangle &= \left[\frac{25}{2}k \cdot \alpha_{-2} + \frac{9}{2}\alpha_{-1} \cdot \alpha_{-1} + \frac{17}{4}(k \cdot \alpha_{-1})^2 \right] |0, k\rangle. \end{aligned} \quad (6.3d)$$

Note that there are two degenerate vector ZNSs, Eq.(6.3a) for type II and Eq.(6.3c) for type I, at mass level $M^2 = 4$. We define D_2 vector ZNS by antisymmetrizing those terms which contain $\alpha_{-1}^\mu \alpha_{-2}^\nu$ in Eq.(6.3a) and Eq.(6.3c) as following

$$|D_2\rangle = \left[\left(\frac{1}{2} k_\mu k_\nu \theta_\lambda + 2\eta_{\mu\nu} \theta_\lambda \right) \alpha_{-1}^\mu \alpha_{-1}^\nu \alpha_{-1}^\lambda + 9k_\mu \theta_\nu \alpha_{-2}^{[\mu} \alpha_{-1}^{\nu]} - 6\theta_\mu \alpha_{-3}^\mu \right] |0, k\rangle, \quad k \cdot \theta = 0. \quad (6.4)$$

Similarly D_1 vector ZNS is defined by symmetrizing those terms which contain $\alpha_{-1}^\mu \alpha_{-2}^\nu$ in Eq.(6.3a) and Eq.(6.3c)

$$|D_1\rangle = \left[\left(\frac{5}{2} k_\mu k_\nu \theta_\lambda + \eta_{\mu\nu} \theta_\lambda \right) \alpha_{-1}^\mu \alpha_{-1}^\nu \alpha_{-1}^\lambda + 9 k_\mu \theta_\nu \alpha_{-2}^{(\mu} \alpha_{-1}^{\nu)} + 6 \theta_\mu \alpha_{-3}^\mu \right] |0, k\rangle, \quad k \cdot \theta = 0. \quad (6.5)$$

In general, *an inter-particle ZNS* can be defined to be $D_2 + \alpha D_1$, where α is an arbitrary constant.

2. ZNS in the helicity basis

In this section, we are going to do the most general spectrum analysis which naturally includes ZNS. We will then *solve* the Virasoro constraints in the helicity basis and recover the ZNS listed above. In particular, this analysis will make it clear how D_2 ZNS in Eq.(6.4) can induce the inter-particle symmetry transformation for two propagating states at the mass level $M^2 = 4$.

We begin our discussion for the mass level $M^2 = 2$. At this mass level, the general expression for the physical states can be written as

$$[\epsilon_{\mu\nu} \alpha_{-1}^\mu \alpha_{-1}^\nu + \epsilon_\mu \alpha_{-2}^\mu] |0, k\rangle. \quad (6.6)$$

In the OCFQ of string theory, physical states satisfy the mass shell condition

$$(L_0 - 1)|phys\rangle = 0 \Rightarrow k^2 = -2; \quad (6.7)$$

and the Virasoro constraints $L_1|phys\rangle = L_2|phys\rangle = 0$ which give

$$\epsilon_\mu = -\epsilon_{\mu\nu} k^\nu, \quad (6.8)$$

$$\eta^{\mu\nu} \epsilon_{\mu\nu} = 2\epsilon_{\mu\nu} k^\mu k^\nu. \quad (6.9)$$

In order to solve for the constraints Eq.(6.8) and Eq.(6.9) in a covariant way, it is convenient to make the following change of basis,

$$e_P \equiv \frac{1}{m}(E, 0, \dots, k) \quad (6.10a)$$

$$e_L \equiv \frac{1}{m}(k, 0, \dots, E) \quad (6.10b)$$

$$e_{T_i} \equiv (0, 0, \dots, 1(\text{i-th spatial direction}), \dots, 0), \quad i = 1, 2, \dots, 24. \quad (6.10c)$$

The 2nd rank tensor $\epsilon_{\mu\nu}$ can be written in the helicity basis Eq.(6.10a) to Eq.(6.10c) as

$$\epsilon_{\mu\nu} = \sum_{A,B} u_{AB} e_\mu^A e_\nu^B, \quad A, B = P, L, T_i. \quad (6.11)$$

In this new representation, the second Virasoro constraint Eq.(6.9) reduces to a simple algebraic relation, and one can solve it

$$u_{PP} = \frac{1}{5} \left(u_{LL} + \sum_{i=1}^{24} u_{T_i T_i} \right). \quad (6.12)$$

In order to perform an irreducible decomposition of the spin-two state into the trace and traceless parts, we define the following variables

$$x \equiv \frac{1}{25} \left(u_{LL} + \sum_{i=1}^{24} u_{T_i T_i} \right), \quad (6.13)$$

$$y \equiv \frac{1}{25} \left(u_{LL} - \frac{1}{24} \sum_{i=1}^{24} u_{T_i T_i} \right). \quad (6.14)$$

We can then write down the complete decompositions of the spin-two polarization tensor as

$$\begin{aligned} \epsilon_{\mu\nu} = & x \left(5e_\mu^P e_\nu^P + e_\mu^L e_\nu^L + \sum_{i=1}^{24} e_\mu^{T_i} e_\nu^{T_i} \right) + y \sum_{i=1}^{24} (e_\mu^L e_\nu^L - e_\mu^{T_i} e_\nu^{T_i}) + \sum_{i,j} (u_{T_i T_j} - \frac{\delta_{ij}}{24} \sum_{l=1}^{24} u_{T_l T_l}) e_\mu^{T_i} e_\nu^{T_j} \\ & + u_{PL} (e_\mu^P e_\nu^L + e_\mu^L e_\nu^P) + \sum_{i=1}^{24} u_{PT_i} (e_\mu^P e_\nu^{T_i} + e_\mu^{T_i} e_\nu^P) + \sum_{i=1}^{24} u_{LT_i} (e_\mu^L e_\nu^{T_i} + e_\mu^{T_i} e_\nu^L). \end{aligned} \quad (6.15)$$

The first Virasoro constraint Eq.(6.8) implies that ϵ_μ vector is not an independent variable, and is related to the spin-two polarization tensor $\epsilon_{\mu\nu}$ as follows

$$\epsilon_\mu = 5\sqrt{2}x e_\mu^P + \sqrt{2}u_{PL} e_\mu^L + \sqrt{2} \sum_{i=1}^{24} u_{PT_i} e_\mu^{T_i}. \quad (6.16)$$

Finally, combining the results of Eq.(6.14), Eq.(6.15) and Eq.(6.16), we get the complete

solution for physical states at mass level $M^2 = 2$

$$[\epsilon_{\mu\nu}\alpha_{-1}^\mu\alpha_{-1}^\nu + \epsilon_\mu\alpha_{-2}^\mu]|0, k\rangle$$

$$= x \left(5\alpha_{-1}^P\alpha_{-1}^P + \alpha_{-1}^L\alpha_{-1}^L + \sum_{i=1}^{24} \alpha_{-1}^{T_i}\alpha_{-1}^{T_i} + 5\sqrt{2}\alpha_{-2}^P \right) |0, k\rangle \quad (6.17)$$

$$+ y \sum_{i=1}^{24} (\alpha_{-1}^L\alpha_{-1}^L - \alpha_{-1}^{T_i}\alpha_{-1}^{T_i}) |0, k\rangle \quad (6.18)$$

$$+ \sum_{i,j} \left(u_{T_i T_j} - \frac{\delta_{ij}}{24} \sum_{l=1}^{24} u_{T_l T_l} \right) \alpha_{-1}^{T_i}\alpha_{-1}^{T_j} |0, k\rangle \quad (6.19)$$

$$+ u_{PL} (2\alpha_{-1}^P\alpha_{-1}^L + \sqrt{2}\alpha_{-2}^L) |0, k\rangle \quad (6.20)$$

$$+ \sum_{i=1}^{24} u_{PT_i} (2\alpha_{-1}^P\alpha_{-1}^{T_i} + \sqrt{2}\alpha_{-2}^{T_i}) |0, k\rangle \quad (6.21)$$

$$+ 2 \sum_{i=1}^{24} u_{LT_i} \alpha_{-1}^L\alpha_{-1}^{T_i} |0, k\rangle, \quad (6.22)$$

where the oscillator creation operators $\alpha_{-1}^P, \alpha_{-1}^L, \alpha_{-1}^{T_i}$, etc., are defined as

$$\alpha_{-n}^A \equiv e_\mu^A \cdot \alpha_{-n}^\mu, \quad n \in N, \quad A = P, L, T_i. \quad (6.23)$$

In comparison with the standard expressions for ZNS in section A, we find that Eq.(6.17), Eq.(6.20) and Eq.(6.21) are identical to the type II singlet and type I vector ZNS for the mass level $M^2 = 2$

$$\text{Eq.(6.17)} = 2x \left[\left(\frac{1}{2}\eta_{\mu\nu} + \frac{3}{2}k_\mu k_\nu \right) \alpha_{-1}^\mu\alpha_{-1}^\nu + \frac{5}{2}k_\mu\alpha_{-2}^\mu \right] |0, k\rangle, \quad (6.24a)$$

$$\text{Eq.(6.20)} = \sqrt{2}u_{PL}[e_\mu^L k_\nu \alpha_{-1}^\mu\alpha_{-1}^\nu + e_\mu^L \alpha_{-2}^\mu] |0, k\rangle, \quad (6.24b)$$

$$\text{Eq.(6.21)} = \sum_{i=1}^{24} \sqrt{2}u_{PT_i} [e_\mu^{T_i} k_\nu \alpha_{-1}^\mu\alpha_{-1}^\nu + e_\mu^{T_i} \alpha_{-2}^\mu] |0, k\rangle. \quad (6.24c)$$

In addition, one can clearly see from our covariant decomposition how ZNS generate gauge transformations on positive-norm states. While a nonzero value for x induces a gauge transformation along the type II singlet ZNS direction, the coefficients u_{PL}, u_{PT_i} parametrize the type I vector gauge transformations with polarization vectors $\theta = e^L$ and $\theta = e^{T_i}$, respectively. Finally, by a simple counting of degrees of freedom, one can identify Eq.(6.18), Eq.(6.19) and Eq.(6.22) as the singlet (1), (traceless) tensor (299), and vector (24) positive-norm states, respectively. These positive-norm states are in a one-to-one correspondence with the degrees of freedom in the light-cone quantization scheme.

We now turn to the analysis of $M^2 = 4$ spectrum. Due to the complexity of our calculations, we shall present the calculations in three steps. We shall first write down all of physical states (including both positive-norm and ZNS) in the simplest gauge choices in the helicity basis. We then calculate the spin-3 state decomposition in the most general gauge choice. Finally, the complete analysis will be given to see how D_2 ZNS in Eq.(6.4) can induce the inter-particle symmetry transformation for two propagating states at the mass level $M^2 = 4$.

a. Physical states in the simplest gauge choices To begin with, let us first analyses the positive-norm states. There are two particles at the mass level $M^2 = 4$, a totally symmetric spin-three particle and an antisymmetric spin-two particle. The canonical representation of the spin-three state is usually chosen as

$$\epsilon_{\mu\nu\lambda} \alpha_{-1}^\mu \alpha_{-1}^\nu \alpha_{-1}^\lambda |0, k\rangle, \quad k^2 = -4, \quad (6.25)$$

where the totally symmetric polarization tensor $\epsilon_{\mu\nu\lambda}$ can be expanded in the helicity basis as

$$\epsilon_{\mu\nu\lambda} = \sum_{A,B,C} \tilde{u}_{ABC} e_\mu^A e_\nu^B e_\lambda^C, \quad A, B, C = P, L, T_i. \quad (6.26)$$

The Virasoro conditions on the polarization tensor can be solved as follows

$$k^\lambda \epsilon_{\mu\nu\lambda} = 0 \Rightarrow \tilde{u}_{PAB} = 0, \quad \forall A, B = P, L, T_i, \quad (6.27)$$

$$\eta^{\nu\lambda} \epsilon_{\mu\nu\lambda} = 0 \Rightarrow \tilde{u}_{LLL} + \sum_i \tilde{u}_{T_i T_i L} = 0, \quad (6.28)$$

$$\tilde{u}_{LLT_i} + \sum_j \tilde{u}_{T_j T_j T_i} = 0.$$

If we choose to keep the minimal number of L components in the expansion coefficients \tilde{u}_{ABC} for the spin-three particle, we get the following canonical decomposition

$$\begin{aligned}
|A(\epsilon)\rangle &\equiv (\epsilon_{\mu\nu\lambda}\alpha_{-1}^\mu\alpha_{-1}^\nu\alpha_{-1}^\lambda)|0, k\rangle = |A(\tilde{u})\rangle \\
&= \sum_i \tilde{u}_{T_i T_i T_i} (\alpha_{-1}^{T_i}\alpha_{-1}^{T_i}\alpha_{-1}^{T_i} - 3\alpha_{-1}^L\alpha_{-1}^L\alpha_{-1}^{T_i})|0, k\rangle \\
&+ \sum_{i \neq j} 3 \tilde{u}_{T_j T_j T_i} (\alpha_{-1}^{T_j}\alpha_{-1}^{T_j}\alpha_{-1}^{T_i} - \alpha_{-1}^L\alpha_{-1}^L\alpha_{-1}^{T_i})|0, k\rangle \\
&+ \sum_{(i \neq j \neq k)} 6 \tilde{u}_{T_i T_j T_k} (\alpha_{-1}^{T_i}\alpha_{-1}^{T_j}\alpha_{-1}^{T_k})|0, k\rangle \\
&+ \sum_i \tilde{u}_{LT_i T_i} (3\alpha_{-1}^L\alpha_{-1}^{T_i}\alpha_{-1}^{T_i} - \alpha_{-1}^L\alpha_{-1}^L\alpha_{-1}^L)|0, k\rangle \\
&+ \sum_{(i \neq j)} 6 \tilde{u}_{LT_i T_j} (\alpha_{-1}^L\alpha_{-1}^{T_i}\alpha_{-1}^{T_j})|0, k\rangle.
\end{aligned} \tag{6.29}$$

It is easy to check that the 2900 independent degrees of freedom of the spin-three particle decompose into $24 + 552 + 2024 + 24 + 276$ in the above representation.

Similarly, for the antisymmetric spin-two particle, we have the following canonical representation

$$\epsilon_{[\mu, \nu]}\alpha_{-1}^\mu\alpha_{-2}^\nu|0, k\rangle. \tag{6.30}$$

Rewriting the polarization tensor $\epsilon_{[\mu, \nu]}$ in the helicity basis

$$\epsilon_{[\mu, \nu]} = \sum_{A, B} v_{[A, B]} e_\mu^A e_\nu^B, \tag{6.31}$$

and solving the Virasoro constraints

$$k^\nu \epsilon_{[\mu, \nu]} = 2v_{[P, L]}e_\mu^L + 2 \sum_{i=1}^{24} v_{[P, T_i]}e_\mu^{T_i} = 0, \tag{6.32}$$

we obtain the following decomposition for the spin-two state

$$\begin{aligned}
|B(\epsilon)\rangle &\equiv \epsilon_{[\mu, \nu]}\alpha_{-1}^\mu\alpha_{-2}^\nu|0, k\rangle = |B(v)\rangle \\
&= \sum_i v_{[L, T_i]} (\alpha_{-1}^L\alpha_{-2}^{T_i} - \alpha_{-1}^{T_i}\alpha_{-2}^L)|0, k\rangle + \sum_{(i \neq j)} v_{[T_i, T_j]} (\alpha_{-1}^{T_i}\alpha_{-2}^{T_j} - \alpha_{-1}^{T_j}\alpha_{-2}^{T_i})|0, k\rangle.
\end{aligned} \tag{6.33}$$

Finally, one can check that the 300 independent degrees of freedom of the spin-two particle decompose into $24 + 276$ in the above expression.

For the ZNS at $M^2 = 4$, we have the following decompositions

1. Spin-two tensor

$$\begin{aligned}
|C(\theta)\rangle &\equiv (k_\lambda \theta_{\mu\nu} \alpha_{-1}^\mu \alpha_{-1}^\nu \alpha_{-1}^\lambda + 2 \theta_{\mu\nu} \alpha_{-1}^\mu \alpha_{-2}^\nu) |0, k\rangle \\
&= \sum_i 2 \theta_{T_i T_i} (\alpha_{-1}^{T_i} \alpha_{-1}^{T_i} \alpha_{-1}^P - \alpha_{-1}^L \alpha_{-1}^L \alpha_{-1}^P + \alpha_{-1}^{T_i} \alpha_{-2}^{T_i} - \alpha_{-1}^L \alpha_{-2}^L) |0, k\rangle \\
&\quad + \sum_{(i \neq j)} 2 \theta_{T_i T_j} (2 \alpha_{-1}^{T_i} \alpha_{-1}^{T_j} \alpha_{-1}^P + \alpha_{-1}^{T_i} \alpha_{-2}^{T_j} + \alpha_{-1}^{T_j} \alpha_{-2}^{T_i}) |0, k\rangle \\
&\quad + \sum_i 2 \theta_{LT_i} (2 \alpha_{-1}^L \alpha_{-1}^{T_i} \alpha_{-1}^P + \alpha_{-1}^L \alpha_{-2}^{T_i} + \alpha_{-1}^{T_i} \alpha_{-2}^L) |0, k\rangle, \tag{6.34}
\end{aligned}$$

where we have solved the Virasoro constraints on the polarization tensor $\theta_{\mu\nu}$

$$\theta_{\mu\nu} = \sum_{A,B} \theta_{AB} e_\mu^A e_\nu^B, \tag{6.35a}$$

$$\eta^{\mu\nu} \theta_{\mu\nu} = -\theta_{PP} + \theta_{LL} + \sum_i \theta_{T_i T_i} = 0, \tag{6.35b}$$

$$k^\nu \theta_{\mu\nu} = -2\theta_{PP} e_\mu^P - 2\theta_{PL} e_\mu^L - 2 \sum_i \theta_{PT_i} e_\mu^{T_i} = 0. \tag{6.35c}$$

The 324 degrees of freedom of on-shell $\theta_{\mu\nu}$ decompose into $24 + 276 + 24$ in Eq.(6.34).

2. Spin-one vector (with polarization vector $\theta \cdot k = 0$, $\theta_\mu = \sum_A \theta_A e_\mu^A$, $A = L, T_i$)

$$\begin{aligned}
|D_1(\theta)\rangle &\equiv \left[\left(\frac{5}{2} k_\mu k_\nu \theta_\lambda + \eta_{\mu\nu} \theta_\lambda \right) \alpha_{-1}^\mu \alpha_{-1}^\nu \alpha_{-1}^\lambda + 9k_{(\mu} \theta_{\nu)} \alpha_{-1}^\mu \alpha_{-2}^\nu + 6\theta_\mu \alpha_{-3}^\mu \right] |0, k\rangle \\
&= \sum_A \theta_A \left[9\alpha_{-1}^P \alpha_{-1}^P \alpha_{-1}^A + \alpha_{-1}^L \alpha_{-1}^L \alpha_{-1}^A + \sum_i \alpha_{-1}^{T_i} \alpha_{-1}^{T_i} \alpha_{-1}^A \right. \\
&\quad \left. + 9(\alpha_{-1}^P \alpha_{-2}^A + \alpha_{-1}^A \alpha_{-2}^P) + 6\alpha_{-3}^A \right] |0, k\rangle. \tag{6.36}
\end{aligned}$$

3. Spin-one vector (with polarization vector $\theta \cdot k = 0$, $\theta_\mu = \sum_A \theta_A e_\mu^A$, $A = L, T_i$)

$$\begin{aligned}
|D_2(\theta)\rangle &\equiv \left[\left(\frac{1}{2} k_\mu k_\nu \theta_\lambda + 2\eta_{\mu\nu} \theta_\lambda \right) \alpha_{-1}^\mu \alpha_{-1}^\nu \alpha_{-1}^\lambda - 9k_{[\mu} \theta_{\nu]} \alpha_{-1}^\mu \alpha_{-2}^\nu - 6\theta_\mu \alpha_{-3}^\mu \right] |0, k\rangle \tag{6.37} \\
&= \sum_A \theta_A \left[2\alpha_{-1}^L \alpha_{-1}^L \alpha_{-1}^A + 2 \sum_j \alpha_{-1}^{T_j} \alpha_{-1}^{T_j} \alpha_{-1}^A - 9(\alpha_{-1}^P \alpha_{-2}^A - \alpha_{-1}^A \alpha_{-2}^P) - 6\alpha_{-3}^A \right] |0, k\rangle. \tag{6.38}
\end{aligned}$$

4. spin-zero singlet

$$\begin{aligned}
|E\rangle &\equiv \left[\left(\frac{17}{4} k_\mu k_\nu k_\lambda + \frac{9}{2} \eta_{\mu\nu} k_\lambda \right) \alpha_{-1}^\mu \alpha_{-1}^\nu \alpha_{-1}^\lambda + (21 k_\mu k_\nu + 9 \eta_{\mu\nu}) \alpha_{-1}^\mu \alpha_{-2}^\nu + 25 k_\mu \alpha_{-3}^\mu \right] |0, k\rangle \\
&= \left[25 (\alpha_{-1}^P \alpha_{-1}^P \alpha_{-1}^P + 3 \alpha_{-1}^P \alpha_{-2}^P + 2 \alpha_{-3}^P) + 9 \alpha_{-1}^L \alpha_{-1}^L \alpha_{-1}^P + 9 \alpha_{-1}^L \alpha_{-2}^L + 9 \sum_i (\alpha_{-1}^{T_i} \alpha_{-1}^{T_i} \alpha_{-1}^P + \alpha_{-1}^{T_i} \alpha_{-2}^{T_i}) \right] |0, k\rangle
\end{aligned} \tag{6.39}$$

b. Spin-three state in the most general gauge choice In this section, we study the most general gauge choice associated with the totally symmetric spin-three state

$$[\varepsilon_{\mu\nu\lambda} \alpha_{-1}^\mu \alpha_{-1}^\nu \alpha_{-1}^\lambda + \varepsilon_{(\mu\nu} \alpha_{-1}^\mu \alpha_{-2}^\nu + \varepsilon_\mu \alpha_{-3}^\mu] |0, k\rangle, \tag{6.40}$$

where Virasoro constraints imply

$$\varepsilon_{(\mu\nu)} = -\frac{3}{2} k^\lambda \varepsilon_{\mu\nu\lambda}, \tag{6.41a}$$

$$\varepsilon_\mu = \frac{1}{2} k^\nu k^\lambda \varepsilon_{\mu\nu\lambda}, \tag{6.41b}$$

$$2\eta^{\mu\nu} \varepsilon_{\mu\nu\lambda} = k^\mu k^\nu \varepsilon_{\mu\nu\lambda}. \tag{6.41c}$$

Eq.(6.41a) and Eq.(6.41b) imply that both $\varepsilon_{(\mu\nu)}$ and ε_μ are not independent variables, and Eq.(6.41c) stands for the constraint on the polarization $\varepsilon_{\mu\nu\lambda}$. In the helicity basis, we define

$$\varepsilon_{\mu\nu\lambda} = \sum_{A,B,C} u_{ABC} e_\mu^A e_\nu^B e_\lambda^C, \quad A, B, C = P, L, T_i. \tag{6.42}$$

Eq.(6.41c) then gives

$$\sum_{A,B} \eta^{AB} u_{ABC} = 2u_{PPC}, \quad A, B, C = P, L, T_i, \tag{6.43}$$

which implies

$$3u_{PPC} - u_{LLC} - \sum_j u_{T_j T_j C} = 0, \quad C = P, L, T_i. \tag{6.44}$$

Eliminating u_{LLP} , u_{LLL} and u_{LLT_i} from above equations, we have the solution for $\varepsilon_{\mu\nu\lambda}$, $\varepsilon_{(\mu\nu)}$ and ε_μ

$$\begin{aligned}
\varepsilon_{\mu\nu\lambda} = & u_{PPP} [e_\mu^P e_\nu^P e_\lambda^P + 3(e_\mu^L e_\nu^L e_\lambda^P + \text{per.})] + u_{PPL} [(e_\mu^P e_\nu^P e_\lambda^L + \text{per.}) + 3e_\mu^L e_\nu^L e_\lambda^L] \\
& + \sum_i u_{PPT_i} [(e_\mu^P e_\nu^P e_\lambda^{T_i} + \text{per.}) + 3(e_\mu^L e_\nu^L e_\lambda^{T_i} + \text{per.})] + \sum_i u_{PT_iT_i} [(e_\mu^P e_\nu^{T_i} e_\lambda^{T_i} + \text{per.}) - (e_\mu^L e_\nu^L e_\lambda^P + \text{per.})] \\
& + \sum_{(i \neq j)} u_{PT_iT_j} [e_\mu^P e_\nu^{T_i} e_\lambda^{T_j} + \text{per.}] + \sum_i u_{PLT_i} [e_\mu^P e_\nu^L e_\lambda^{T_i} + \text{per.}] + \sum_i u_{LT_iT_i} [(e_\mu^L e_\nu^{T_i} e_\lambda^{T_i} + \text{per.}) - e_\mu^L e_\nu^L e_\lambda^L] \\
& + \sum_{(i \neq j)} u_{LT_iT_j} [e_\mu^L e_\nu^{T_i} e_\lambda^{T_j} + \text{per.}] + \sum_i u_{T_iT_iT_i} [e_\mu^{T_i} e_\nu^{T_i} e_\lambda^{T_i} - (e_\mu^L e_\nu^L e_\lambda^{T_i} + \text{per.})] \\
& + \sum_{i \neq j} u_{T_jT_jT_i} [(e_\mu^{T_j} e_\nu^{T_j} e_\lambda^{T_i} + \text{per.}) - (e_\mu^L e_\nu^L e_\lambda^{T_i} + \text{per.})] \\
& + \sum_{(i \neq j \neq k)} u_{T_iT_jT_k} [e_\mu^{T_i} e_\nu^{T_j} e_\lambda^{T_k} + \text{per.}], \tag{6.45}
\end{aligned}$$

$$\begin{aligned}
\frac{1}{3}\varepsilon_{(\mu\nu)} = & u_{PPP}(e_\mu^P e_\nu^P + 3e_\mu^L e_\nu^L) + u_{PPL}(e_\mu^P e_\nu^L + e_\mu^L e_\nu^P) + \sum_i u_{PPT_i}(e_\mu^P e_\nu^{T_i} + e_\mu^{T_i} e_\nu^P) \\
& + \sum_i u_{PLT_i}(e_\mu^L e_\nu^{T_i} + e_\mu^{T_i} e_\nu^L) + \sum_i u_{PT_iT_i}(e_\mu^{T_i} e_\nu^{T_i} - e_\mu^L e_\nu^L) + \sum_{(i \neq j)} u_{PT_iT_j}(e_\mu^{T_i} e_\nu^{T_j} + e_\nu^{T_j} e_\mu^{T_j}), \tag{6.46}
\end{aligned}$$

$$\frac{1}{2}\varepsilon_\mu = [u_{PPP} e_\mu^P + u_{PPL} e_\mu^L + \sum_i u_{PPT_i} e_\mu^{T_i}]. \tag{6.47}$$

Putting all these polarizations back to the general form of physical states Eq.(6.40), we get

$$\begin{aligned}
& [\varepsilon_{\mu\nu\lambda}\alpha_{-1}^\mu\alpha_{-1}^\nu\alpha_{-1}^\lambda + \varepsilon_{(\mu\nu)}\alpha_{-1}^\mu\alpha_{-2}^\nu + \varepsilon_\mu\alpha_{-3}^\mu]|0, k\rangle = |A(\tilde{u})\rangle + |C(\theta)\rangle \\
& + \left[\frac{1}{9}(u_{LLL} + \sum_i u_{T_iT_iL}) \right] |D_1(e^L)\rangle \\
& + \sum_i \left[\frac{1}{9}(u_{LLT_i} + \sum_j u_{T_jT_jT_i}) \right] |D_1(e^{T_i})\rangle \\
& + \frac{1}{75} \left[u_{LLP} + \sum_i u_{PT_iT_i} \right] |E\rangle. \tag{6.48}
\end{aligned}$$

For the first two terms on the right hand side of Eq.(6.48), we need to make the following replacements. For the positive-norm state $|A(\tilde{u})\rangle$ in Eq.(6.29)

$$\begin{aligned}
\tilde{u}_{T_iT_iT_i} & \rightarrow u_{T_iT_iT_i} - \frac{1}{3}u_{PPT_i}, & \tilde{u}_{T_jT_jT_i} & \rightarrow u_{T_jT_jT_i} - \frac{1}{9}u_{PPT_i}, \\
\tilde{u}_{T_iT_jT_k} & \rightarrow u_{T_iT_jT_k} & \tilde{u}_{LT_iT_i} & \rightarrow u_{LT_iT_i} - \frac{1}{9}u_{PPL}, & \tilde{u}_{LT_iT_j} & \rightarrow u_{LT_iT_j}. \tag{6.49}
\end{aligned}$$

For the spin-two ZNS $|C(\theta)\rangle$ in Eq.(6.34), the replacement is given by

$$2\theta_{LT_i} \rightarrow 3u_{PLT_i}, \quad 2\theta_{T_iT_j} \rightarrow 3u_{PT_iT_j}, \text{ for } i \neq j, \quad 2\theta_{T_iT_i} \rightarrow 3(u_{PT_iT_i} - \frac{3}{25}u_{PPP}). \quad (6.50)$$

It is important to note that for the spin-three gauge multiplet, only spin-two, singlet and D_1 vector ZNS appear in the decomposition Eq.(6.48). In the next section, we will see how one can include the missing D_2 ZNS in the analysis.

c. Complete spectrum analysis and the D_2 ZNS After all these preparations, we are ready for a complete analysis of the most general decomposition of physical states at $M^2 = 4$. The most general form of physical states at this mass level are given by

$$[\epsilon_{\mu\nu\lambda}\alpha_{-1}^\mu\alpha_{-1}^\nu\alpha_{-1}^\lambda + \epsilon_{(\mu\nu)}\alpha_{-1}^\mu\alpha_{-2}^\nu + \epsilon_{[\mu\nu]}\alpha_{-1}^\mu\alpha_{-2}^\nu + \epsilon_\mu\alpha_{-3}^\mu]|0, k\rangle. \quad (6.51)$$

The Virasoro constraints are

$$\epsilon_{(\mu\nu)} = -\frac{3}{2}k^\lambda\epsilon_{\mu\nu\lambda}, \quad (6.52a)$$

$$-k^\nu\epsilon_{[\mu\nu]} + 3\epsilon_\mu = \frac{3}{2}k^\nu k^\lambda\epsilon_{\mu\nu\lambda}, \quad (6.52b)$$

$$2k^\nu\epsilon_{[\mu\nu]} + 3\epsilon_\mu = 3(k^\nu k^\lambda - \eta^{\nu\lambda})\epsilon_{\mu\nu\lambda}. \quad (6.52c)$$

The solutions to Eq.(6.52b) and Eq.(6.52c) are given by

$$k^\nu\epsilon_{[\mu\nu]} = \left(\frac{1}{2}k^\nu k^\lambda - \eta^{\nu\lambda}\right)\epsilon_{\mu\nu\lambda}, \quad (6.53)$$

$$3\epsilon_\mu = (2k^\nu k^\lambda - \eta^{\nu\lambda})\epsilon_{\mu\nu\lambda}. \quad (6.54)$$

In contrast to the previous discussion Eq.(6.41a) and Eq.(6.41b) where both $\epsilon_{(\mu\nu)}$ and ϵ_μ are completely fixed by the leading spin-three polarization tensor $\epsilon_{\mu\nu\lambda}$, we now have a new contribution from $k^\nu\epsilon_{[\mu\nu]}$. It will become clear that this extra term includes the inter-particle ZNS D_2 , Eq.(6.37) or Eq.(6.38). Furthermore, it should be clear that the antisymmetric spin-two positive-norm physical states are defined by requiring $\epsilon_{\mu\nu\lambda} = \epsilon_{(\mu\nu)} = 0$ and $\epsilon_\mu = k^\nu\epsilon_{[\mu\nu]} = 0$. In the following, for the sake of clarity, we shall focus on the effects of the new contribution induced by the $\epsilon_{[\mu\nu]}$ only.

The two independent polarization tensors of the most general representation for physical states Eq.(6.51) are given in the helicity basis by

$$\epsilon_{\mu\nu\lambda} = \sum_{ABC} U_{ABC} e_\mu^A e_\nu^B e_\lambda^C, \quad A, B, C = P, L, T_i; \quad (6.55)$$

$$\epsilon_{[\mu\nu]} = \sum_{A,B} V_{[AB]} e_\mu^A e_\nu^B. \quad (6.56)$$

The Virasoro constraint Eq.(6.53) demands that

$$3U_{PPP} - U_{LLP} - \sum_i U_{PT_i T_i} = 0, \quad (6.57a)$$

$$3U_{PPL} - U_{LLL} - \sum_i U_{LT_i T_i} = 2V_{[PL]}, \quad (6.57b)$$

$$3U_{PPT_i} - U_{LLT_i} - \sum_j U_{T_j T_j T_i} = 2V_{[PT_i]}. \quad (6.57c)$$

In contrast to Eq.(6.44), the solution to the above equations become

$$U_{PPL} = U_{PPL}^{(1)} + U_{PPL}^{(2)}, \text{ where } U_{PPL}^{(1)} = \frac{1}{3}(U_{LLL} + \sum_i U_{T_i T_i L}), \quad U_{PPL}^{(2)} = \frac{2}{3}V_{[PL]}; \quad (6.58)$$

$$U_{PPT_i} = U_{PPT_i}^{(1)} + U_{PPT_i}^{(2)}, \text{ where } U_{PPT_i}^{(1)} = \frac{1}{3}(U_{LLT_i} + \sum_j U_{T_j T_j T_i}), \quad U_{PPT_i}^{(2)} = \frac{2}{3}V_{[PT_i]}. \quad (6.59)$$

It is clear from the expressions above that only $U_{PPL}^{(2)}$ and $U_{PPT_i}^{(2)}$ give new contributions to our previous analysis in the last section, so we can simply write down all these new terms

$$\delta\epsilon_{\mu\nu\lambda} = \frac{2}{3}[V_{[PL]}(e_\mu^P e_\nu^P e_\lambda^L + \text{per.}) + \sum_i V_{[PT_i]}(e_\mu^P e_\nu^P e_\lambda^{T_i} + \text{per.})], \quad (6.60a)$$

$$\begin{aligned} \delta\epsilon_{[\mu\nu]} &= V_{[PL]}(e_\mu^P e_\nu^L - \text{per.}) + \sum_i V_{[PT_i]}(e_\mu^P e_\nu^{T_i} - \text{per.}) \\ &\quad + \sum_i V_{[T_i L]}(e_\mu^{T_i} e_\nu^L - \text{per.}) + \sum_{i \neq j} V_{[T_j T_i]}(e_\mu^{T_j} e_\nu^{T_i} - \text{per.}), \end{aligned} \quad (6.60b)$$

$$\delta\epsilon_{(\mu\nu)} = 2[V_{[PL]}(e_\mu^P e_\nu^L + \text{per.}) + \sum_i V_{[PT_i]}(e_\mu^P e_\nu^{T_i} + \text{per.})], \quad (6.60c)$$

$$\delta\epsilon_\mu = 2[V_{[PL]}e_\mu^L + \sum_i V_{[PT_i]}e_\mu^{T_i}]. \quad (6.60d)$$

Finally, the complete decomposition of physical states Eq.(6.51) in the helicity basis becomes

$$\begin{aligned} &[\epsilon_{\mu\nu\lambda}\alpha_{-1}^\mu\alpha_{-1}^\nu\alpha_{-1}^\lambda + \epsilon_{(\mu\nu)}\alpha_{-1}^\mu\alpha_{-2}^\nu + \epsilon_{[\mu\nu]}\alpha_{-1}^\mu\alpha_{-2}^\nu + \epsilon_\mu\alpha_{-3}^\mu]|0, k\rangle \\ &= |A(U_{CBA})\rangle + |B(V_{[T_i A]})\rangle + |C(U_{PBA})\rangle \end{aligned} \quad (6.61a)$$

$$+ \sum_{A=L, T_i} \left[\frac{1}{9}(U_{LLA} + \sum_i U_{T_i T_i A}) \right] |D_1(e^A)\rangle \quad (6.61b)$$

$$- \frac{1}{9} \sum_{A=L, T_i} V_{[PA]} |D'_2(e^A)\rangle \quad (6.61c)$$

$$+ \frac{1}{75} [U_{LLP} + \sum_i U_{PT_i T_i}] |E\rangle. \quad (6.61d)$$

In Eq.(6.61a), $|A(U_{CBA})\rangle$ is given by Eq.(6.29) with \tilde{u}_{CBA} given by Eq.(6.49) and we have replaced u by U on the r.h.s. of Eq.(6.49). The antisymmetric spin-two positive-norm state $|B(V_{[T_i A]})\rangle$ is given by Eq.(6.33) and we have replaced v by V in Eq.(6.33). Finally, $|C(U_{PBA})\rangle$ is given by Eq.(6.34) with θ given by Eq.(6.50) and we have replaced u by U on the r.h.s. of Eq.(6.50). In Eq.(6.61c), $|D'_2(e^A)\rangle \equiv |D_2(e^A)\rangle - 2|D_1(e^A)\rangle$ is the inter-particle ZNS introduced in the end of section A with $\alpha = -2$. Note that the value of α is a choice of convention fixed by the parametrization of the polarizations. It can always be adjusted to be zero. In view of Eq.(6.58) and Eq.(6.59), we see that one can use either $V_{[PA]}$ or $U_{PPA}^{(2)}$ ($A = L, T_i$) to represent the polarization of the $|D'_2(e^A)\rangle$ inter-particle ZNS.

We conclude that once we turn on the antisymmetric spin-two positive-norm state in the general representation of physical states Eq.(6.51), it is naturally accompanied by the D'_2 inter-particle ZNS. The polarization of the D'_2 inter-particle ZNS can be represented by either $V_{[PA]}$ or $U_{PPA}^{(2)}$ ($A = L, T_i$) in Eq.(6.55) and Eq.(6.56). Thus this inter-particle ZNS will generate an inter-particle symmetry transformation in the σ -model calculation considered in chapter I. Note that, in contrast to the high-energy symmetry of Gross, this symmetry is valid to all orders in α' .

B. Light-cone ZNS in DDF construction

In the usual light-cone quantization of bosonic string theory, one solves the Virasoro constraints to get rid of two string coordinates X^\pm . Only 24 string coordinates α_n^i , $i = 1, \dots, 24$, remain, and there are no ZNS in the spectrum. However, there existed another similar quantization scheme, the DDF quantization, which did *include the ZNS* in the spectrum. In the light-cone DDF quantization of open bosonic string [127–129], one constructs transverse physical states with discrete momenta

$$p^\mu = p_0^\mu - Nk_0^\mu = (1, 0, \dots, -1 + N), \quad (6.62)$$

where $X^\pm \equiv \frac{1}{\sqrt{2}}(X^0 \pm X^{25})$ and $p^+ = 1$, $p^- = -1 + N$. In Eq.(6.62), $M^2 = -p^2 = 2(N - 1)$ and $p_0^\mu \equiv (1, 0, \dots, -1)$, $k_0^\mu \equiv (0, 0, \dots, -1)$, respectively. All other states can be reached by Lorentz transformations. The DDF operators are given by [127–129]

$$A_n^i = \frac{1}{2\pi} \int_0^{2\pi} \dot{X}^i(\tau) e^{inX^+(\tau)} d\tau, \quad i = 1, \dots, 24, \quad (6.63)$$

where the massless vertex operator $V^i(nk_0, \tau) = \dot{X}^i(\tau)e^{inX^+(\tau)}$ is a primary field with conformal dimension one, and is periodic in the worldsheet time τ if one chooses $k^\mu = nk_0^\mu$ with $n \in \mathbb{Z}$. It is then easy to show that

$$[L_m, A_n^i] = 0, \quad (6.64)$$

$$[A_m^i, A_n^j] = m\delta_{ij}\delta_{m+n}. \quad (6.65)$$

In addition to sharing the same algebra, Eq.(6.65), with string coordinates α_n^i , the DDF operators A_n^i possess a nicer property Eq.(6.64), which enables us to easily write down a general formula for the positive-norm physical states as following

$$(A_{-1}^j)^{i_1}(A_{-2}^k)^{i_2} \dots (A_{-m}^l)^{i_m} |0, p_0\rangle, \quad i_r \in \mathbb{Z}, \quad (6.66)$$

where $|0, p_0\rangle$ is the tachyon ground state and $N = \sum_{r=1}^m r i_r$ is the level of the state. Historically, DDF operators were used to prove no-ghost (negative-norm states) theorem for $D = 26$ string theory. Here we are going to use them to analyse ZNS. It turns out that ZNS can be generated by

$$\tilde{A}_n^- = A_n^- - \sum_{m=1}^{\infty} \sum_{i=1}^{D-2} : A_m^i A_{n-m}^i :, \quad (6.67)$$

where A_n^- is given by

$$A_n^- = \frac{1}{2\pi} \int_0^{2\pi} \left[: \dot{X}^- e^{inX^+} : - \frac{1}{2} in \frac{d}{d\tau} (\log \dot{X}^+) e^{inX^+} \right] d\tau. \quad (6.68)$$

It can be shown that \tilde{A}_n^- commute with L_m and satisfy the following algebra

$$[\tilde{A}_m^-, A_n^i] = 0, \quad (6.69)$$

$$[\tilde{A}_m^-, \tilde{A}_n^-] = (m-n)\tilde{A}_{m+n}^- + \frac{26-D}{12} m^3 \delta_{m+n}. \quad (6.70)$$

Eq.(6.65), Eq.(6.69) and Eq.(6.70) constitute the spectrum generating algebra for the open bosonic string including ZNS. The ground state $|0, p_0\rangle \equiv |0\rangle$ satisfies the following conditions

$$A_n^i |0\rangle = \tilde{A}_n^- |0\rangle = 0, \quad n > 0, \quad (6.71)$$

$$\tilde{A}_0^- |0\rangle = -\frac{26-D}{24}, \quad A_0^i |0\rangle = 0. \quad (6.72)$$

We are now ready to construct ZNS in the DDF formalism.

1. $M^2 = 0$: One has only one scalar $\tilde{A}_{-1}^- |0\rangle$, which has zero-norm for any D .

2. $M^2 = 2$: One has a light-cone vector $A_{-1}^i \tilde{A}_{-1}^- |0\rangle$, which has zero-norm for any D , and two scalars, whose norms are calculated to be

$$\| (a \tilde{A}_{-1}^- \tilde{A}_{-1}^- + b \tilde{A}_{-2}^-) |0\rangle \| = \frac{26-D}{2} b^2. \quad (6.73)$$

For $b = 0$, one has a "pure type I" ZNS, $\tilde{A}_{-1}^- \tilde{A}_{-1}^- |0\rangle$, which has zero-norm for any D . By combining with the light-cone vector $A_{-1}^i \tilde{A}_{-1}^- |0\rangle$, one obtains a vector ZNS with 25 degrees of freedom, which corresponds to Eq.(6.2b) in the OCFQ approach. For $b \neq 0$, one obtains a type II scalar ZNS for $D = 26$, which corresponds to Eq.(6.2a) in the OCFQ approach.

3. $M^2 = 4$:

I. A spin-two tensor $A_{-1}^i A_{-1}^j \tilde{A}_{-1}^- |0\rangle$, which has zero-norm for any D .

II. Three light-cone vectors, whose norms are calculated to be

$$\| (a A_{-1}^i \tilde{A}_{-1}^- \tilde{A}_{-1}^- + b A_{-2}^i \tilde{A}_{-1}^- + c A_{-1}^i \tilde{A}_{-2}^-) |0\rangle \| = \frac{26-D}{2} c^2. \quad (6.74)$$

III. Three scalars, whose norms are calculated to be

$$\| (d \tilde{A}_{-1}^- \tilde{A}_{-1}^- \tilde{A}_{-1}^- + e \tilde{A}_{-1}^- \tilde{A}_{-2}^- + f \tilde{A}_{-3}^-) |0\rangle \| = 2(26-D)(e+f)^2. \quad (6.75)$$

For $c = 0$ in Eq.(6.74), one has two "pure type I" light-cone vector ZNS. For $e + f = 0$ in Eq.6.75), one has two "pure type I" scalar ZNS. One of the two type I light-cone vectors, when combining with the spin-two state in I, gives the type I spin-two tensor which corresponds to Eq.(6.3b) in the OCFQ approach. The other type I light-cone vector, when combining with one of the two type I scalar, gives the type I vector ZNS which corresponds to Eq.(6.3c) in the OCFQ approach. The other type I scalar corresponds to Eq.(6.3d). Finally, for $c \neq 0$ and $e + f \neq 0$, one obtains the type II vector ZNS for $D = 26$, which corresponds to Eq.(6.3a) in the OCFQ approach. It is easy to see that a special linear combination of b and e will give the inter-particle vector ZNS which corresponds to the inter-particle D_2 ZNS in Eq.(6.4). This completes the analysis of ZNS for $M^2 = 4$.

Note that the exact mapping of ZNS in the light-cone DDF formalism and the OCFQ approach depends on the exact relation between operators $(\tilde{A}_n^-, A_n^i, L_n)$ and α_n^μ , which has not been worked out in the literature.

C. BRST ZNS in WSFT

In this section, we calculate BRST ZNS in the formulation of WSFT. In addition, we apply the results to demonstrate that off-shell gauge transformations of WSFT are indeed identical to the on-shell stringy gauge symmetries generated by two types of ZNS in the generalized massive σ -model approach [6, 8] of string theory. In section I.D [14], the background ghost transformations in the gauge transformations of WSFT [16] were shown to correspond, in a one-to-one manner, to the lifting of on-shell conditions of ZNS in the OCFQ approach. Here we go one step further to demonstrate the correspondence of stringy symmetries induced by ZNS in OCFQ and BRST approaches.

Cubic string field theory is defined on a disk with the action

$$S = -\frac{1}{g_0} \left(\frac{1}{2} \int \Phi * Q_B \Phi + \frac{1}{3} \int \Phi * \Phi * \Phi \right), \quad (6.76)$$

where Q_B is the BRST charge

$$Q_B = \sum_{n=-\infty}^{\infty} L_{-n}^m c_n + \sum_{m,n=-\infty}^{\infty} \frac{m-n}{2} : c_m c_n b_{-m-n} : -c, \quad (6.77)$$

and Φ is the string field with ghost number 1 and b, c are conformal ghosts. Since the ghost number of vacuum on a disk is -3 , the total ghost number of this action is 0 as expected. The string field can be expanded as

$$\Phi = \sum_{k,m,n} A_{\mu\dots,k\dots m\dots n\dots}(x) \alpha_k^\mu \cdots b_m \cdots c_n \cdots |\Omega\rangle, \quad (6.78)$$

where the string ground states $|\Omega\rangle$ are

$$|\Omega\rangle = c_1 |0\rangle. \quad (6.79)$$

The gauge transformation for string field can be written as

$$\delta\Phi = Q_B \Lambda + g (\Phi * \Lambda - \Lambda * \Phi). \quad (6.80)$$

where Λ is the a string field with ghost number 0.

For the purpose of discussion in this chapter, we are going to consider the linearized gauge transformation

$$\delta\Phi = Q_B \Lambda, \quad (6.81)$$

where $Q_B \Lambda$ is just the off-shell ZNS. In the following, we will explicitly show that the solutions of Eq.(6.81) are in one-to-one correspondence to the ZNS obtained in OCFQ approach in section VI.A level by level for the first several mass levels.

There is no ZNS in the lowest string mass level with $M^2 = -2$, so our analysis will start with the mass level of $M^2 = 0$.

$M^2 = 0$:

The string field can be expanded as

$$\Phi = \{iA_\mu(x) \alpha_{-1}^\mu + \alpha(x) b_{-1} c_0\} |\Omega\rangle, \quad (6.82)$$

$$\Lambda = \{\epsilon^0(x) b_{-1}\} |\Omega\rangle. \quad (6.83)$$

The gauge transformation is then

$$Q_B \Lambda = \left\{ -\frac{1}{2} \alpha_0^2 \epsilon^0 b_{-1} c_0 + \epsilon^0 \alpha_0 \cdot \alpha_{-1} \right\} |\Omega\rangle. \quad (6.84)$$

The nilpotency of BRST charge Q_B gives

$$Q_B^2 \Lambda = 0, \quad (6.85)$$

which can be easily checked to be valid for any D . Thus Eq.(6.84) can be interpreted as a type I ZNS. To compare it with the ZNS obtained in OCFQ approach in section II, we need to reduce our Hilbert space by removing the ghosts states. In particular, the coefficients of terms with ghost operators must vanish. For the state in Eq.(6.84), it is

$$\alpha_0^2 \epsilon^0 = 0, \quad (6.86)$$

which give the on-shell condition $k^2 = 0$ and the following ZNS

$$Q_B \Lambda = \epsilon^0 \alpha_0 \cdot \alpha_{-1} |\Omega\rangle. \quad (6.87)$$

This is the same as the scalar ZNS obtained in OCFQ approach.

$M^2 = 2$:

The string fields expansion are

$$\begin{aligned} \Phi = \{ & -B_{\mu\nu}(x) \alpha_{-1}^\mu \alpha_{-1}^\nu + iB_\mu(x) \alpha_{-2}^\mu \\ & + i\beta_\mu(x) \alpha_{-1}^\mu b_{-1} c_0 + \beta^0(x) b_{-2} c_0 + \beta^1(x) b_{-1} c_{-1} \} |\Omega\rangle, \end{aligned} \quad (6.88)$$

$$\Lambda = \{ i\epsilon_\mu^0(x) \alpha_{-1}^\mu b_{-1} + \epsilon^1(x) b_{-2} \} |\Omega\rangle. \quad (6.89)$$

The off-shell ZNS are calculated to be

$$\begin{aligned}
Q_B \Lambda = & \left\{ \left(i\alpha_{0\mu} \epsilon_\nu^0 + \frac{1}{2} \epsilon^1 \eta_{\mu\nu} \right) \alpha_{-1}^\mu \alpha_{-1}^\nu + (i\epsilon^0 + \epsilon^1 \alpha_0) \cdot \alpha_{-2} \right. \\
& - i\frac{1}{2} (\alpha_0^2 + 2) (\epsilon^0 \cdot \alpha_{-1}) b_{-1} c_0 - \frac{1}{2} (\alpha_0^2 + 2) \epsilon^1 b_{-2} c_0 \\
& \left. - (i\epsilon^0 \cdot \alpha_0 + 3\epsilon^1) b_{-1} c_{-1} \right\} |\Omega\rangle.
\end{aligned} \tag{6.90}$$

Nilpotency condition requires

$$Q_B^2 \Lambda = \frac{D-26}{2} \epsilon^1 c_{-2} |\Omega\rangle = 0. \tag{6.91}$$

There are two solutions of Eq.(6.91), which correspond to the type I and type II ZNS, respectively.

1. Type I: in this case D is not restricted to the critical string dimension in Eq.(6.91), i.e. $D \neq 26$. Thus

$$\epsilon^1 = 0. \tag{6.92}$$

The no-ghost conditions of Eq.(6.90) lead to the on-shell constraints

$$\alpha_0^2 + 2 = 0, \tag{6.93}$$

$$\epsilon^0 \cdot \alpha_0 = 0. \tag{6.94}$$

The off-shell ZNS in Eq.(6.90) then reduces to an on-shell vector ZNS

$$Q_B \Lambda = i \left\{ (\epsilon^0 \cdot \alpha_{-1}) (\alpha_0 \cdot \alpha_{-1}) + \epsilon^0 \cdot \alpha_{-2} \right\} |\Omega\rangle \tag{6.95}$$

2. Type II: in this case D is restricted to the critical string dimension, i.e. $D = 26$. Then ϵ^1 can be arbitrary constant. The no-ghost conditions then lead to the on-shell constraints

$$\alpha_0^2 + 2 = 0, \tag{6.96}$$

$$i\epsilon^0 \cdot \alpha_0 + 3\epsilon^1 = 0. \tag{6.97}$$

The second condition can be solved by a special solution

$$\epsilon_\mu^0 = -\frac{3i}{2} \epsilon^1 \alpha_{0\mu}, \tag{6.98}$$

which leads to an on-shell scalar ZNS

$$Q_B \Lambda = \epsilon^1 \left\{ \frac{3}{2} (\alpha_0 \cdot \alpha_{-1})^2 + \frac{1}{2} (\alpha_{-1} \cdot \alpha_{-1}) + \frac{5}{2} (\alpha_0 \cdot \alpha_{-2}) \right\} |\Omega\rangle \tag{6.99}$$

Again, up to a constant factor, the ZNS Eq.(6.95) and Eq.(6.99) are the same as Eq.(6.2b) and Eq.(6.2a) calculated in the OCFQ approach.

$M^2 = 4$:

The string fields are expanded as

$$\begin{aligned} \Phi = & \left\{ -iC_{\mu\nu\lambda}(x) \alpha_{-1}^\mu \alpha_{-1}^\nu \alpha_{-1}^\lambda - C_{\mu\nu}(x) \alpha_{-2}^\mu \alpha_{-1}^\nu + iC_\mu(x) \alpha_{-3}^\mu \right. \\ & - \gamma_{\mu\nu}(x) \alpha_{-1}^\mu \alpha_{-1}^\nu b_{-1} c_0 + i\gamma_\mu^0(x) \alpha_{-1}^\mu b_{-2} c_0 + i\gamma_\mu^1(x) \alpha_{-1}^\mu b_{-1} c_{-1} \\ & \left. + i\gamma_\mu^2(x) \alpha_{-2}^\mu b_{-1} c_0 + \gamma^0(x) b_{-3} c_0 + \gamma^1(x) b_{-2} c_{-1} + \gamma^2(x) b_{-1} c_{-2} \right\} |\Omega\rangle, \end{aligned} \quad (6.100)$$

$$\begin{aligned} \Lambda = & \left\{ -\epsilon_{\mu\nu}(x) \alpha_{-1}^\mu \alpha_{-1}^\nu b_{-1} + i\epsilon_\mu^1(x) \alpha_{-2}^\mu b_{-1} |\Omega\rangle \right. \\ & \left. + i\epsilon_\mu^2(x) \alpha_{-1}^\mu b_{-2} + \epsilon^2(x) b_{-3} + \epsilon^3(x) b_{-1} b_{-2} c_0 \right\} |\Omega\rangle. \end{aligned} \quad (6.101)$$

The off-shell ZNS are

$$\begin{aligned} Q_B \Lambda = & \left\{ \left(-\alpha_{0(\mu} \epsilon_{\nu\lambda)} + \frac{i}{2} \epsilon_{(\mu}^2 \eta_{\nu\lambda)} \right) \alpha_{-1}^\mu \alpha_{-1}^\nu \alpha_{-1}^\lambda + (i\alpha_{0\mu} \epsilon_\nu^2 + i\alpha_{0\nu} \epsilon_\mu^1 - 2\epsilon_{\mu\nu} + \epsilon^2 \eta_{\mu\nu}) \alpha_{-2}^\mu \alpha_{-1}^\nu \right. \\ & + (\alpha_{0\mu} \epsilon^2 + 2i\epsilon_\mu^1 + i\epsilon_\mu^2) \alpha_{-3}^\mu + \left[\frac{1}{2} (\alpha_0^2 + 4) \epsilon_{\mu\nu} + \frac{1}{2} \epsilon^3 \eta_{\mu\nu} \right] \alpha_{-1}^\mu \alpha_{-1}^\nu b_{-1} c_0 \\ & + \left[-\frac{i}{2} (\alpha_0^2 + 4) \epsilon_\mu^2 - \alpha_{0\mu} \epsilon^3 \right] \alpha_{-1}^\mu b_{-2} c_0 + (2\alpha_0^\nu \epsilon_{\nu\mu} - 2i\epsilon_\mu^1 - 3i\epsilon_\mu^2) \alpha_{-1}^\mu b_{-1} c_{-1} \\ & + \left[-\frac{i}{2} (\alpha_0^2 + 4) \epsilon_\mu^1 + \alpha_{0\mu} \epsilon^3 \right] \alpha_{-2}^\mu b_{-1} c_0 + \left[-\frac{1}{2} (\alpha_0^2 + 4) \epsilon^2 - \epsilon^3 \right] b_{-3} c_0 \\ & \left. + (-i\alpha_0^\mu \epsilon_\mu^2 - 4\epsilon^2 - 2\epsilon^3) b_{-2} c_{-1} + (-2i\alpha_0^\mu \epsilon_\mu^1 - 5\epsilon^2 + 4\epsilon^3 + \epsilon_\mu^\mu) b_{-1} c_{-2} \right\} |\Omega\rangle. \end{aligned} \quad (6.102)$$

Nilpotency condition requires

$$Q_B^2 \Lambda = (D - 26) \left[\frac{i}{2} \epsilon_\mu^2 \alpha_{-1}^\mu c_{-2} + 2\epsilon^2 c_{-3} - \frac{1}{2} \epsilon^3 b_{-1} c_{-2} c_0 \right] = 0. \quad (6.103)$$

Similarly, we classify the solutions of Eq.(6.103) by type I and type II in the following:

1. Type I: $D \neq 26$. This leads to

$$\epsilon^2 = \epsilon^3 = \epsilon_\mu^2 = 0, \quad (6.104)$$

The no-ghost conditions lead to the on-shell constraints

$$\alpha_0^2 + 4 = 0, \quad (6.105a)$$

$$\alpha_0^\nu \epsilon_{\nu\mu} - i\epsilon_\mu^1 = 0, \quad (6.105b)$$

$$-2i(\alpha_0 \cdot \epsilon^1) + \epsilon_\mu^\mu = 0. \quad (6.105c)$$

There are three independent solutions to the above equations, which correspond to the three type I on-shell ZNS:

- Tensor ZNS

$$\epsilon_\mu^1 = 0, \quad \alpha_0^\nu \epsilon_{\mu\nu} = 0, \quad \epsilon_\mu^\mu = 0, \quad (6.106)$$

$$Q_B \Lambda = - \left\{ \alpha_{0\mu} \epsilon_{\nu\lambda} \alpha_{-1}^\mu \alpha_{-1}^\nu \alpha_{-1}^\lambda + 2 \epsilon_{\mu\nu} \alpha_{-2}^\mu \alpha_{-1}^\nu \right\} |\Omega\rangle. \quad (6.107)$$

- Vector ZNS

$$\alpha_0 \cdot \epsilon^1 = 0, \quad \epsilon_{\mu\nu} = -\frac{i}{4} (\alpha_{0\nu} \epsilon_\mu^1 + \alpha_{0\mu} \epsilon_\nu^1). \quad (6.108)$$

$$Q_B \Lambda = \left\{ i \frac{1}{2} (\alpha_0 \cdot \alpha_{-1})^2 (\epsilon^1 \cdot \alpha_{-1}) + 2i (\epsilon^1 \cdot \alpha_{-3}) \right. \\ \left. + \frac{3}{2} (\alpha_0 \cdot \alpha_{-1}) (\epsilon^1 \cdot \alpha_{-2}) + \frac{1}{2} (\alpha_0 \cdot \alpha_{-2}) (\epsilon^1 \cdot \alpha_{-1}) \right\} |\Omega\rangle. \quad (6.109)$$

- Scalar ZNS

$$\epsilon_\mu^1 = \frac{i(D-1)}{9} \theta \alpha_{0\mu}, \quad \epsilon_{\mu\nu} = \theta \eta_{\mu\nu} + \frac{(8+D)}{36} \theta \alpha_{0\mu} \alpha_{0\nu}. \quad (6.110)$$

$$Q_B \Lambda = -\frac{2}{9} \theta \left\{ \frac{(8+D)}{8} (\alpha_0 \cdot \alpha_{-1})^3 + \frac{9}{2} (\alpha_0 \cdot \alpha_{-1}) (\alpha_{-1} \cdot \alpha_{-1}) + 9 (\alpha_{-1} \cdot \alpha_{-2}) \right. \\ \left. + \frac{3(D+2)}{4} (\alpha_0 \cdot \alpha_{-1}) (\alpha_0 \cdot \alpha_{-2}) + (D-1) (\alpha_0 \cdot \alpha_{-3}) \right\} |\Omega\rangle. \quad (6.111)$$

If we set $D = 26$, then

$$Q_B \Lambda = -\frac{2}{9} \theta \left\{ \frac{17}{4} (\alpha_0 \cdot \alpha_{-1})^3 + \frac{9}{2} (\alpha_0 \cdot \alpha_{-1}) (\alpha_{-1} \cdot \alpha_{-1}) + 9 (\alpha_{-1} \cdot \alpha_{-2}) \right. \\ \left. + 21 (\alpha_0 \cdot \alpha_{-1}) (\alpha_0 \cdot \alpha_{-2}) + 25 (\alpha_0 \cdot \alpha_{-3}) \right\} |\Omega\rangle, \quad (6.112)$$

where θ is an arbitrary constant.

2. Type II: $D = 26$ in Eq.(6.103), and ϵ^2, ϵ^3 and ϵ_μ^2 are arbitrary constants. The no-ghost conditions lead to the on-shell constraints

$$\alpha_0^2 + 4 = 0, \quad (6.113a)$$

$$\epsilon^3 = 0, \quad (6.113b)$$

$$2\alpha_0^\nu \epsilon_{\nu\mu} - 2i\epsilon_\mu^1 - 3i\epsilon_\mu^2 = 0, \quad (6.113c)$$

$$i\alpha_0^\mu \epsilon_\mu^2 + 4\epsilon^2 = 0, \quad (6.113d)$$

$$-2i\alpha_0^\mu \epsilon_\mu^1 - 5\epsilon^2 + \epsilon_\mu^\mu = 0. \quad (6.113e)$$

A special solution of above equations is

$$\epsilon^2 = -\frac{i}{4} (\alpha_0 \cdot \epsilon^2) = 0, \quad (6.114a)$$

$$\epsilon_{\mu\nu} = -C (\alpha_{0\mu} \epsilon_\nu^2 + \alpha_{0\nu} \epsilon_\mu^2), \quad (6.114b)$$

$$\epsilon_\mu^1 = \frac{8iC - 3}{2} \epsilon_\mu^2, \quad (6.114c)$$

which gives an on-shell vector ZNS

$$\begin{aligned} Q_B \Lambda = i \bigg\{ & (8iC - 2) (\epsilon^2 \cdot \alpha_{-3}) + \frac{1}{2} (\alpha_{-1} \cdot \alpha_{-1}) (\epsilon^2 \cdot \alpha_{-1}) \\ & + (2iC + 1) (\alpha_0 \cdot \alpha_{-2}) (\epsilon^2 \cdot \alpha_{-1}) + 2iC (\alpha_0 \cdot \alpha_{-1})^2 (\epsilon^2 \cdot \alpha_{-1}) \\ & + \frac{12iC - 3}{2} (\alpha_0 \cdot \alpha_{-1}) (\epsilon^2 \cdot \alpha_{-2}) \bigg\} |\Omega\rangle. \end{aligned} \quad (6.115)$$

For a special value of $C = -3i/4$, Eq.(6.115) becomes

$$\begin{aligned} Q_B \Lambda = i \bigg\{ & 4 (\epsilon^2 \cdot \alpha_{-3}) + \frac{1}{2} (\alpha_{-1} \cdot \alpha_{-1}) (\epsilon^2 \cdot \alpha_{-1}) + \frac{5}{2} (\alpha_0 \cdot \alpha_{-2}) (\epsilon^2 \cdot \alpha_{-1}) \\ & + \frac{3}{2} (\alpha_0 \cdot \alpha_{-1})^2 (\epsilon^2 \cdot \alpha_{-1}) + 3 (\alpha_0 \cdot \alpha_{-1}) (\epsilon^2 \cdot \alpha_{-2}) \bigg\} |\Omega\rangle. \end{aligned} \quad (6.116)$$

Up to a constant factor, ZNS in Eq.(6.107), Eq.(6.109), Eq.(6.112) and Eq.(6.116) are exactly the same as Eq.(6.3b), Eq.(6.3c), Eq.(6.3d) and Eq.(6.3a) calculated in the OCFQ approach. In addition, it can be checked that for $C = -5i/8$ and $-i/16$ in Eq.(6.115), one gets D_1 and D_2 ZNS of OCFQ approach in Eq.(6.5) and Eq.(6.4) respectively.

In section I.D [14], the background ghost transformations in the gauge transformations of WSFT [16] were shown to correspond, in a one-to-one manner, to the lifting of on-shell conditions of ZNS in the OCFQ approach. For the rest of this section, we are going to go one step further and apply the results calculated above to demonstrate that off-shell gauge transformations of WSFT are indeed identical to the on-shell stringy gauge symmetries generated by two types of ZNS in the generalized massive σ -model approach [6, 8] of string theory. For the mass level $M^2 = 2$, by using Eq.(6.88) and Eq.(6.89), the linearized gauge transformation of WSFT in Eq.(6.81)

gives

$$\delta B_{\mu\nu} = -\partial_{(\mu}\epsilon_{\nu)}^0 - \frac{1}{2}\epsilon^1\eta_{\mu\nu}, \quad (6.117a)$$

$$\delta B_\mu = -\partial_\mu\epsilon^1 + \frac{1}{2}\epsilon_\mu^0, \quad (6.117b)$$

$$\delta\beta_\mu = \frac{1}{2}(\partial^2 - 2)\epsilon_\mu^0, \quad (6.117c)$$

$$\delta\beta^0 = \frac{1}{2}(\partial^2 - 2)\epsilon^1, \quad (6.117d)$$

$$\delta\beta^1 = -\partial^\mu\epsilon_\mu^0 - 3\epsilon^1 \quad (6.117e)$$

For the type I gauge transformation induced by ZNS in Eq.(6.95), one can use Eq.(6.92) to Eq.(6.94) to eliminate the background ghost transformations Eq.(6.117c) to Eq.(6.117e). Finally, conditions of worldsheet conformal invariance in the presence of weak background fields [6, 8] can be used to express B_μ in terms of $B_{\mu\nu}$, and one ends up with the following on-shell gauge transformation by Eq.(6.117a)

$$\delta B_{\mu\nu} = \partial_{(\mu}\epsilon_{\nu)}^0; \quad \partial^\mu\epsilon_\mu^0 = 0, \quad (\partial^2 - 2)\epsilon_\mu^0 = 0. \quad (6.118)$$

Similarly, one can apply the same procedure to type II ZNS in Eq.(6.99), and derive the following type II gauge transformation

$$\delta B_{\mu\nu} = \frac{3}{2}\partial_\mu\partial_\nu\epsilon^1 - \frac{1}{2}\eta_{\mu\nu}\epsilon^1, \quad (\partial^2 - 2)\epsilon^1 = 0. \quad (6.119)$$

Eq.(6.118) and Eq.(6.119) are consistent with the massive σ -model calculation in the OCFQ string theory in.

For the mass level $M^2 = 4$, by using Eq.(6.100) and Eq.(6.101), the linearized gauge

transformation of WSFT in Eq.(6.81) gives

$$\delta C_{\mu\nu\lambda} = -\partial_{(\mu}\epsilon_{\nu\lambda)}^0 - \frac{1}{2}\epsilon_{(\mu}^2\eta_{\mu\nu)}, \quad (6.120a)$$

$$\delta C_{[\mu\nu]} = -\partial_{[\nu}\epsilon_{\mu]}^1 - \partial_{[\mu}\epsilon_{\nu]}^2, \quad (6.120b)$$

$$\delta C_{(\mu\nu)} = -\partial_{(\nu}\epsilon_{\mu)}^1 - \partial_{(\mu}\epsilon_{\nu)}^2 + 2\epsilon_{\mu\nu}^0 - \epsilon^2\eta_{\mu\nu}, \quad (6.120c)$$

$$\delta C_\mu = -\partial_\mu\epsilon^2 + 2\epsilon_\mu^1 + \epsilon_\mu^2, \quad (6.120d)$$

$$\delta\gamma_{\mu\nu} = \frac{1}{2}(\partial^2 - 4)\epsilon_{\mu\nu}^0 - \frac{1}{2}\epsilon^3\eta_{\mu\nu}, \quad (6.120e)$$

$$\delta\gamma_\mu^0 = \frac{1}{2}(\partial^2 - 4)\epsilon_\mu^2 + \partial_\mu\epsilon^3, \quad (6.120f)$$

$$\delta\gamma_\mu^1 = -2\partial^\nu\epsilon_{\nu\mu}^0 - 2\epsilon_\mu^1 - 3\epsilon_\mu^2, \quad (6.120g)$$

$$\delta\gamma_\mu^2 = \frac{1}{2}(\partial^2 - 4)\epsilon_\mu^1 - \partial_\mu\epsilon^3, \quad (6.120h)$$

$$\delta\gamma^0 = \frac{1}{2}(\partial^2 - 4)\epsilon^2 - \epsilon^3, \quad (6.120i)$$

$$\delta\gamma^1 = -\partial^\mu\epsilon_\mu^2 - 4\epsilon^2 - 2\epsilon^3, \quad (6.120j)$$

$$\delta\gamma^2 = -2\partial^\mu\epsilon_\mu^1 - 5\epsilon^2 + 4\epsilon^3 + \epsilon_\mu^{0\mu}. \quad (6.120k)$$

For the gauge transformation induced by D_2 ZNS in Eq.(6.116), for example, one can use Eq.(6.113a) to Eq.(6.114c) with $C = -i/16$ to eliminate Eq.(6.120e) to Eq.(6.120k). One can then use the fact that background fields $C_{(\mu\nu)}$ and C_μ are gauge artifacts of $C_{\mu\nu\lambda}$ in the σ -model calculation, and deduce from Eq.(6.120a) to Eq.(6.120d) the inter-particle symmetry transformation

$$\delta C_{\mu\nu\lambda} = \frac{1}{2}\partial_{(\mu}\partial_{\nu}\epsilon_{\lambda)}^{(D_2)} - 2\eta_{(\mu\nu}\epsilon_{\lambda)}^{(D_2)}, \quad \delta C_{[\mu\nu]} = 9\partial_{[\mu}\epsilon_{\nu]}^{(D_2)}, \quad (6.121)$$

where $\partial^\lambda\epsilon_\lambda^{(D_2)} = 0$, $(\partial^2 - 4)\epsilon_\lambda^{(D_2)} = 0$. The other three gauge transformations corresponding to three other ZNS, the spin-two, D_1 , and scalar can be similarly constructed from Eq.(6.120a) to Eq.(6.120k). One gets

$$\delta C_{\mu\nu\lambda} = \partial_{(\mu}\epsilon_{\nu\lambda)}; \quad \partial^\mu\epsilon_{\mu\nu} = 0, \quad (\partial^2 - 4)\epsilon_{\mu\nu} = 0, \quad (6.122)$$

$$\delta C_{\mu\nu\lambda} = \frac{5}{2}\partial_{(\mu}\partial_{\nu}\epsilon_{\lambda)}^{(D_1)} - \eta_{(\mu\nu}\epsilon_{\lambda)}^{(D_1)}; \quad \partial^\lambda\epsilon_\lambda^{(D_1)} = 0, \quad (\partial^2 - 4)\epsilon_\lambda^{(D_1)} = 0, \quad (6.123)$$

$$\delta C_{\mu\nu\lambda} = \frac{17}{4}\partial_\mu\partial_\nu\partial_\lambda\theta - \frac{9}{2}\eta_{(\mu\nu}\theta_{\lambda)}; \quad (\partial^2 - 4)\theta = 0. \quad (6.124)$$

Eq.(6.121) to Eq.(6.124) are exactly the same as those calculated by the generalized massive σ -model approach of string theory [6, 8].

We thus have shown in this section that off-shell gauge transformations of WSFT are identical to the on-shell stringy gauge symmetries generated by two types of ZNS in the OCFQ string theory. The high energy limit of these stringy gauge symmetries generated by ZNS was recently used to fix the proportionality constants among high energy scattering amplitudes of different string states conjectured by Gross [3, 4]. Based on the ZNS calculations in [26–29] and the calculations in this section, we thus have related gauge symmetry of WSFT [16] to the high energy stringy symmetry conjectured by Gross [1–5].

In conclusion of this chapter, we have calculated ZNS in the OCFQ string, the light-cone DDF string and the off-shell BRST string theories. In the OCFQ string, we have solved the Virasoro constraints for all physical states (including ZNS) in the helicity basis. Much attention was paid to discuss the inter-particle ZNS at the mass level $M^2 = 4$. We found that one can use polarization of either one of the two positive-norm states to represent the polarization of the inter-particle ZNS. This justified why one can derive the inter-particle symmetry transformation for the two massive modes in the weak field massive σ -model calculation [6, 8].

In the light-cone DDF string, one can easily write down the general formula for all ZNS in the spectrum. We have identified type I and Type II ZNS up to the mass level $M^2 = 4$. An analysis for the general mass levels should be easy to generalize.

Finally, we have calculated off-shell ZNS in the WSFT. After imposing the no ghost conditions, we can recover two types of on-shell ZNS in the OCFQ string. We then show that off-shell gauge transformations of WSFT are identical to the on-shell stringy gauge symmetries generated by two types of ZNS in the generalized massive σ -model approach of string theory. Based on these ZNS calculations, we thus have related gauge symmetry of WSFT [16] to the high energy stringy symmetry of Gross [3, 4].

VII. HARD CLOSED STRING SCATTERINGS, KLT AND STRING BCJ RELATIONS

In this chapter, we generalize the calculations in chapter V to high energy closed string scattering amplitudes [35]. We will find that the methods of decoupling of high energy ZNS and the high energy Virasoro constraints, which were adopted in chapter V to calculate the ratios among high energy open string scattering amplitudes of different string states, persist

for the case of closed string. The result is simply the tensor product of two pieces of open string ratios of high energy scattering amplitudes.

However, we clarify the previous saddle-point calculation for high energy open string scattering amplitudes and claim that only (t, u) channel of the amplitudes is suitable for saddle-point calculation. We then discuss three evidences to show that saddle-point calculation for high energy closed string scattering amplitudes is not reliable. By using the relation of tree-level closed and open string scattering amplitudes of Kawai, Lewellen and Tye (KLT) [43, 130], we calculate the tree-level high energy closed string scattering amplitudes for *arbitrary* mass levels. For the case of high energy closed string four-tachyon amplitude, our result differs from the previous one of Gross and Mende [1, 2], which is NOT consistent with KLT formula, by an oscillating factor. See also [131, 132]. One interesting application of this result is the string BCJ relations [37, 40, 123–125] which will be discussed in section D.

A. Decoupling of high energy ZNS

In this section, we calculate the ratios among high energy closed string scattering amplitudes of different string states by the decoupling of high energy closed string ZNS. Since the calculation is similar to that of open string in chapter V, we will, for simplicity, work on the first massive level $M^2 = 8(N - 1) = 8$ only. At this mass level, the corresponding open string Ward identities are ($M^2 = 2$ for open string, $\alpha'_{\text{closed}} = 4\alpha'_{\text{open}} = 2$) [32]

$$k_\mu \theta_\nu \mathcal{T}^{\mu\nu} + \theta_\mu \mathcal{T}^\mu = 0, \quad (7.1a)$$

$$\left(\frac{3}{2} k_\mu k_\nu + \frac{1}{2} \eta_{\mu\nu} \right) \mathcal{T}^{\mu\nu} + \frac{5}{2} k_\mu \mathcal{T}^\mu = 0, \quad (7.1b)$$

where θ_ν is a transverse vector. In Eq.(7.1a) and Eq.(7.1b), we have chosen, say, the second vertex $V_2(k_2)$ to be the vertex operators constructed from ZNS and $k_\mu \equiv k_{2\mu}$. The other three vertices can be any string states. Note that Eq.(7.1a) is the type I Ward identity while Eq.(7.1b) is the type II Ward identity which is valid only at $D = 26$. The high energy limits of Eq.(7.1a) and Eq.(7.1b) were calculated to be

$$M \mathcal{T}_{TP}^{3 \rightarrow 1} + \mathcal{T}_T^1 = 0, \quad (7.2a)$$

$$M \mathcal{T}_{LL}^{4 \rightarrow 2} + \mathcal{T}_L^2 = 0, \quad (7.2b)$$

$$3M^2 \mathcal{T}_{LL}^{4 \rightarrow 2} + \mathcal{T}_{TT}^2 + 5M \mathcal{T}_L^2 = 0. \quad (7.2c)$$

Note that since \mathcal{T}_{TP}^1 is of subleading order in energy, in general $\mathcal{T}_{TP}^1 \neq \mathcal{T}_{TL}^1$. A simple calculation of Eq.(7.2a) to Eq.(7.2c) shows that [32]

$$\mathcal{T}_{TP}^1 : \mathcal{T}_T^1 = 1 : -\sqrt{2} = 1 : -M. \quad (7.3)$$

$$\mathcal{T}_{TT}^2 : \mathcal{T}_{LL}^2 : \mathcal{T}_L^2 = 4 : 1 : -\sqrt{2} = 2M^2 : 1 : -M. \quad (7.4)$$

It is interesting to see that, in addition to the leading order amplitudes in Eq.(7.4), the subleading order amplitudes in Eq.(7.3) are also proportional to each other. This does not seem to happen at higher mass level.

We are now back to the closed string calculation. The OCFQ closed string spectrum at this mass level are $(\square\square + \square + \bullet) \otimes (\square\square + \square + \bullet)'$. In addition to the spin-four positive-norm state $\square\square \otimes \square\square'$, one has 8 ZNS, each of which gives a Ward identity. In the high energy limit, we have $\theta^{\mu\nu} = e_L^\mu e_L^\nu - e_T^\mu e_T^\nu$ or $\theta^{\mu\nu} = e_L^\mu e_T^\nu + e_T^\mu e_L^\nu$, $\theta^\mu = e_L^\mu$ or e_T^μ and one replace $\eta_{\mu\nu}$ by $e_T^\mu e_T^\nu$. In the following, we list only high energy Ward identities which relate amplitudes with even-energy power in the high energy expansion :

1. $\square\square \otimes \square'$:

$$M(\mathcal{T}_{LL,LL} - \mathcal{T}_{TT,LL}) + \mathcal{T}_{LL,L} - \mathcal{T}_{TT,L} = 0, \quad (7.5)$$

$$M\mathcal{T}_{LT,PT} + \mathcal{T}_{LT,T} = 0. \quad (7.6)$$

2. $\square\square \otimes \bullet'$:

$$3M^2(\mathcal{T}_{LL,LL} - \mathcal{T}_{TT,LL}) + (\mathcal{T}_{LL,TT} - \mathcal{T}_{TT,TT}) + 5M(\mathcal{T}_{LL,L} - \mathcal{T}_{TT,L}) = 0. \quad (7.7)$$

3. $\square \otimes \square\square'$:

$$M(\mathcal{T}_{LL,LL} - \mathcal{T}_{LL,TT}) + \mathcal{T}_{L,LL} - \mathcal{T}_{L,TT} = 0, \quad (7.8)$$

$$M\mathcal{T}_{PT,LT} + \mathcal{T}_{T,LT} = 0. \quad (7.9)$$

4. $\square \otimes \square'$:

$$M^2\mathcal{T}_{LL,LL} + M\mathcal{T}_{LL,L} + M\mathcal{T}_{L,LL} + \mathcal{T}_{L,L} = 0, \quad (7.10)$$

$$M^2\mathcal{T}_{PT,PT} + M\mathcal{T}_{PT,T} + M\mathcal{T}_{T,PT} + \mathcal{T}_{T,T} = 0. \quad (7.11)$$

5. $\square \otimes \bullet'$:

$$3M^3\mathcal{T}_{LL,LL} + M\mathcal{T}_{LL,TT} + 5M^2\mathcal{T}_{LL,L} + 3M^2\mathcal{T}_{L,LL} + \mathcal{T}_{L,TT} + 5M^2\mathcal{T}_{L,L} = 0. \quad (7.12)$$

6. $\bullet \otimes \square\square'$:

$$3M^2(\mathcal{T}_{LL,LL} - \mathcal{T}_{LL,TT}) + (\mathcal{T}_{TT,LL} - \mathcal{T}_{TT,TT}) + 5M(\mathcal{T}_{L,LL} - \mathcal{T}_{L,TT}) = 0. \quad (7.13)$$

7. $\bullet \otimes \square'$:

$$3M^3\mathcal{T}_{LL,LL} + M\mathcal{T}_{TT,LL} + 5M^2\mathcal{T}_{L,LL} + 3M^2\mathcal{T}_{LL,L} + \mathcal{T}_{TT,L} + 5M^2\mathcal{T}_{L,L} = 0. \quad (7.14)$$

8. $\bullet \otimes \bullet'$:

$$\begin{aligned} & 9M^4\mathcal{T}_{LL,LL} + 3M^2\mathcal{T}_{LL,TT} + 3M^2\mathcal{T}_{TT,LL} + 15M^3\mathcal{T}_{LL,L} \\ & + 15M^3\mathcal{T}_{L,LL} + 5M\mathcal{T}_{TT,L} + 5M\mathcal{T}_{L,TT} + 25M^2\mathcal{T}_{L,L} + \mathcal{T}_{TT,TT} = 0. \end{aligned} \quad (7.15)$$

Those Ward identities which relate amplitudes with odd-energy power in the high energy expansion are omitted as they are subleading order in energy. The mass M in Eq.(7.5) to Eq.(7.15) should now be interpreted as the closed string mass $M^2 = 8$. Eq.(7.6),Eq.(7.9) and Eq.(7.11) are subleading order amplitudes, and one can then solve the other 8 equations to give the ratios

$$\begin{aligned} & \mathcal{T}_{TT,TT} : \mathcal{T}_{TT,LL} : \mathcal{T}_{LL,TT} : \mathcal{T}_{LL,LL} : \mathcal{T}_{TT,L} : \mathcal{T}_{L,TT} : \mathcal{T}_{LL,L} : \mathcal{T}_{L,LL} : \mathcal{T}_{L,L} \\ & = 1 : \frac{1}{2M^2} : \frac{1}{2M^2} : \frac{1}{4M^4} : -\frac{1}{2M} : -\frac{1}{2M} : -\frac{1}{4M^3} : -\frac{1}{4M^3} : \frac{1}{4M^2}. \end{aligned} \quad (7.16)$$

Eq.(7.16) is exactly the tensor product of two pieces of open string ratios calculated in Eq.(7.4).

B. Virasoro constraints

We consider the mass level $M^2 = 8$. The most general state is

$$\begin{aligned} |2\rangle &= \left\{ \frac{1}{2!} \boxed{\mu_1^1 \mu_2^1} \alpha_{-1}^{\mu_1^1} \alpha_{-1}^{\mu_2^1} + \frac{1}{2} \boxed{\mu_1^2} \alpha_{-2}^{\mu_1^2} \right\} \otimes \left\{ \frac{1}{2!} \boxed{\tilde{\mu}_1^1 \tilde{\mu}_2^1} \tilde{\alpha}_{-1}^{\tilde{\mu}_1^1} \tilde{\alpha}_{-1}^{\tilde{\mu}_2^1} + \frac{1}{2} \boxed{\tilde{\mu}_1^2} \tilde{\alpha}_{-2}^{\tilde{\mu}_1^2} \right\} |0, k\rangle \\ &= \frac{1}{4} \left\{ \boxed{\mu_1^1 \mu_2^1} \alpha_{-1}^{\mu_1^1} \alpha_{-1}^{\mu_2^1} + \boxed{\mu_1^2} \alpha_{-2}^{\mu_1^2} \right\} \otimes \left\{ \boxed{\tilde{\mu}_1^1 \tilde{\mu}_2^1} \tilde{\alpha}_{-1}^{\tilde{\mu}_1^1} \tilde{\alpha}_{-1}^{\tilde{\mu}_2^1} + \boxed{\tilde{\mu}_1^2} \tilde{\alpha}_{-2}^{\tilde{\mu}_1^2} \right\} |0, k\rangle. \end{aligned} \quad (7.17)$$

The Virasoro constraints are

$$L_1 |2\rangle \sim \left\{ k^{\mu_1^1} \boxed{\mu_1^1 \mu_2^1} \alpha_{-1}^{\mu_1^1} \alpha_{-1}^{\mu_2^1} + \boxed{\mu_1^2} \alpha_{-2}^{\mu_1^2} \right\} \otimes \left\{ \boxed{\tilde{\mu}_1^1 \tilde{\mu}_2^1} \tilde{\alpha}_{-1}^{\tilde{\mu}_1^1} \tilde{\alpha}_{-1}^{\tilde{\mu}_2^1} + \boxed{\tilde{\mu}_1^2} \tilde{\alpha}_{-2}^{\tilde{\mu}_1^2} \right\} = 0, \quad (7.18a)$$

$$\tilde{L}_1 |2\rangle \sim \left\{ \boxed{\mu_1^1 \mu_2^1} \alpha_{-1}^{\mu_1^1} \alpha_{-1}^{\mu_2^1} + \boxed{\mu_1^2} \alpha_{-2}^{\mu_1^2} \right\} \otimes \left\{ k^{\mu_1^1} \boxed{\tilde{\mu}_1^1 \tilde{\mu}_2^1} \tilde{\alpha}_{-1}^{\tilde{\mu}_1^1} \tilde{\alpha}_{-1}^{\tilde{\mu}_2^1} + \boxed{\tilde{\mu}_1^2} \tilde{\alpha}_{-2}^{\tilde{\mu}_1^2} \right\} = 0, \quad (7.18b)$$

$$L_2 |2\rangle \sim \left\{ \boxed{\mu_1^1 \mu_2^1} \eta^{\mu_1^1 \mu_2^1} + 2k^{\mu_1^2} \boxed{\mu_1^2} \right\} \otimes \left\{ \boxed{\tilde{\mu}_1^1 \tilde{\mu}_2^1} \tilde{\alpha}_{-1}^{\tilde{\mu}_1^1} \tilde{\alpha}_{-1}^{\tilde{\mu}_2^1} + \boxed{\tilde{\mu}_1^2} \tilde{\alpha}_{-2}^{\tilde{\mu}_1^2} \right\} = 0, \quad (7.18c)$$

$$\tilde{L}_2 |2\rangle \sim \left\{ \boxed{\mu_1^1 \mu_2^1} \alpha_{-1}^{\mu_1^1} \alpha_{-1}^{\mu_2^1} + \boxed{\mu_1^2} \alpha_{-2}^{\mu_1^2} \right\} \otimes \left\{ \boxed{\tilde{\mu}_1^1 \tilde{\mu}_2^1} \eta^{\tilde{\mu}_1^1 \tilde{\mu}_2^1} + 2k^{\tilde{\mu}_1^2} \boxed{\tilde{\mu}_1^2} \right\} = 0. \quad (7.18d)$$

Taking the high energy limit in the above equations by letting $(\mu_i, \nu_i) \rightarrow (L, T)$, and

$$k^{\mu_i} \rightarrow Me^L, \eta^{\mu_1 \mu_2} \rightarrow e^T e^T, \quad (7.19)$$

we obtain

$$\left\{ M \begin{bmatrix} L \\ \mu \end{bmatrix} + \begin{bmatrix} \mu \end{bmatrix} \right\} \alpha_{-1}^\mu \otimes \left\{ \begin{bmatrix} \tilde{\mu}_1^1 \\ \tilde{\mu}_2^1 \end{bmatrix} \tilde{\alpha}_{-1}^{\tilde{\mu}_1^1} \tilde{\alpha}_{-1}^{\tilde{\mu}_2^1} + \begin{bmatrix} \tilde{\mu}_1^2 \\ \tilde{\mu}_2^2 \end{bmatrix} \tilde{\alpha}_{-1}^{\tilde{\mu}_1^2} \tilde{\alpha}_{-1}^{\tilde{\mu}_2^2} \right\} = 0, \quad (7.20a)$$

$$\left\{ \begin{bmatrix} \mu_1^1 \\ \mu_2^1 \end{bmatrix} \alpha_{-1}^{\mu_1^1} \alpha_{-1}^{\mu_2^1} + \begin{bmatrix} \mu_1^2 \\ \mu_2^2 \end{bmatrix} \alpha_{-1}^{\mu_1^2} \alpha_{-1}^{\mu_2^2} \right\} \otimes \left\{ M \begin{bmatrix} L \\ \tilde{\mu} \end{bmatrix} + \begin{bmatrix} \tilde{\mu} \end{bmatrix} \right\} \tilde{\alpha}_{-1}^{\tilde{\mu}} = 0, \quad (7.20b)$$

$$\left\{ \begin{bmatrix} T \\ T \end{bmatrix} + 2M \begin{bmatrix} L \end{bmatrix} \right\} \otimes \left\{ \begin{bmatrix} \tilde{\mu}_1^1 \\ \tilde{\mu}_2^1 \end{bmatrix} \tilde{\alpha}_{-1}^{\tilde{\mu}_1^1} \tilde{\alpha}_{-1}^{\tilde{\mu}_2^1} + \begin{bmatrix} \tilde{\mu}_1^2 \\ \tilde{\mu}_2^2 \end{bmatrix} \tilde{\alpha}_{-1}^{\tilde{\mu}_1^2} \tilde{\alpha}_{-1}^{\tilde{\mu}_2^2} \right\} = 0, \quad (7.20c)$$

$$\left\{ \begin{bmatrix} \mu_1^1 \\ \mu_2^1 \end{bmatrix} \alpha_{-1}^{\mu_1^1} \alpha_{-1}^{\mu_2^1} + \begin{bmatrix} \mu_1^2 \\ \mu_2^2 \end{bmatrix} \alpha_{-1}^{\mu_1^2} \alpha_{-1}^{\mu_2^2} \right\} \otimes \left\{ \begin{bmatrix} T \\ T \end{bmatrix} + 2M \begin{bmatrix} L \end{bmatrix} \right\} = 0, \quad (7.20d)$$

which lead to the following equations

$$\left\{ M \begin{bmatrix} L \\ \mu \end{bmatrix} + \begin{bmatrix} \mu \end{bmatrix} \right\} \otimes \begin{bmatrix} \tilde{\mu}_1^1 \\ \tilde{\mu}_2^1 \end{bmatrix} = 0, \quad (7.21a)$$

$$\left\{ M \begin{bmatrix} L \\ \mu \end{bmatrix} + \begin{bmatrix} \mu \end{bmatrix} \right\} \otimes \begin{bmatrix} \tilde{\mu}_1^2 \end{bmatrix} = 0, \quad (7.21b)$$

$$\begin{bmatrix} \mu_1^1 \\ \mu_2^1 \end{bmatrix} \otimes \left\{ M \begin{bmatrix} L \\ \tilde{\mu} \end{bmatrix} + \begin{bmatrix} \tilde{\mu} \end{bmatrix} \right\} = 0, \quad (7.21c)$$

$$\begin{bmatrix} \mu_1^2 \end{bmatrix} \otimes \left\{ M \begin{bmatrix} L \\ \tilde{\mu} \end{bmatrix} + \begin{bmatrix} \tilde{\mu} \end{bmatrix} \right\} = 0, \quad (7.21d)$$

$$\left\{ \begin{bmatrix} T \\ T \end{bmatrix} + 2M \begin{bmatrix} L \end{bmatrix} \right\} \otimes \begin{bmatrix} \tilde{\mu}_1^1 \\ \tilde{\mu}_2^1 \end{bmatrix} = 0, \quad (7.21e)$$

$$\left\{ \begin{bmatrix} T \\ T \end{bmatrix} + 2M \begin{bmatrix} L \end{bmatrix} \right\} \otimes \begin{bmatrix} \tilde{\mu}_1^2 \end{bmatrix} = 0, \quad (7.21f)$$

$$\begin{bmatrix} \mu_1^1 \\ \mu_2^1 \end{bmatrix} \otimes \left\{ \begin{bmatrix} T \\ T \end{bmatrix} + 2M \begin{bmatrix} L \end{bmatrix} \right\} = 0, \quad (7.21g)$$

$$\begin{bmatrix} \mu_1^2 \end{bmatrix} \otimes \left\{ \begin{bmatrix} T \\ T \end{bmatrix} + 2M \begin{bmatrix} L \end{bmatrix} \right\} = 0. \quad (7.21h)$$

The remaining indices $\mu, \tilde{\mu}$ in the above equations can be set to be T or L , and we obtain

$$M \begin{bmatrix} L \\ L \end{bmatrix} \otimes \begin{bmatrix} L \\ L \end{bmatrix} + \begin{bmatrix} L \end{bmatrix} \otimes \begin{bmatrix} L \\ L \end{bmatrix} = 0, \quad (7.22a)$$

$$M \begin{bmatrix} L \\ L \end{bmatrix} \otimes \begin{bmatrix} T \\ T \end{bmatrix} + \begin{bmatrix} L \end{bmatrix} \otimes \begin{bmatrix} T \\ T \end{bmatrix} = 0, \quad (7.22b)$$

$$M \begin{bmatrix} T \\ L \end{bmatrix} \otimes \begin{bmatrix} T \\ L \end{bmatrix} + \begin{bmatrix} T \end{bmatrix} \otimes \begin{bmatrix} T \\ L \end{bmatrix} = 0, \quad (7.22c)$$

$$M \begin{bmatrix} L \\ L \end{bmatrix} \otimes \begin{bmatrix} L \end{bmatrix} + \begin{bmatrix} L \end{bmatrix} \otimes \begin{bmatrix} L \end{bmatrix} = 0, \quad (7.23a)$$

$$M \begin{bmatrix} T \\ L \end{bmatrix} \otimes \begin{bmatrix} T \end{bmatrix} + \begin{bmatrix} T \end{bmatrix} \otimes \begin{bmatrix} T \end{bmatrix} = 0, \quad (7.23b)$$

$$M \begin{array}{|c|c|} \hline L & L \\ \hline \end{array} \otimes \begin{array}{|c|c|} \hline L & L \\ \hline \end{array} + \begin{array}{|c|c|} \hline L & L \\ \hline \end{array} \otimes \begin{array}{|c|} \hline L \\ \hline \end{array} = 0, \quad (7.24a)$$

$$M \begin{array}{|c|c|} \hline T & T \\ \hline \end{array} \otimes \begin{array}{|c|c|} \hline L & L \\ \hline \end{array} + \begin{array}{|c|c|} \hline T & T \\ \hline \end{array} \otimes \begin{array}{|c|} \hline L \\ \hline \end{array} = 0, \quad (7.24b)$$

$$M \begin{array}{|c|c|} \hline T & L \\ \hline \end{array} \otimes \begin{array}{|c|c|} \hline T & L \\ \hline \end{array} + \begin{array}{|c|c|} \hline T & L \\ \hline \end{array} \otimes \begin{array}{|c|} \hline T \\ \hline \end{array} = 0, \quad (7.24c)$$

$$M \begin{array}{|c|} \hline L \\ \hline \end{array} \otimes \begin{array}{|c|c|} \hline L & L \\ \hline \end{array} + \begin{array}{|c|} \hline L \\ \hline \end{array} \otimes \begin{array}{|c|} \hline L \\ \hline \end{array} = 0, \quad (7.25a)$$

$$M \begin{array}{|c|} \hline T \\ \hline \end{array} \otimes \begin{array}{|c|c|} \hline T & L \\ \hline \end{array} + \begin{array}{|c|} \hline T \\ \hline \end{array} \otimes \begin{array}{|c|} \hline T \\ \hline \end{array} = 0, \quad (7.25b)$$

$$\begin{array}{|c|c|} \hline T & T \\ \hline \end{array} \otimes \begin{array}{|c|c|} \hline L & L \\ \hline \end{array} + 2M \begin{array}{|c|} \hline L \\ \hline \end{array} \otimes \begin{array}{|c|c|} \hline L & L \\ \hline \end{array} = 0, \quad (7.26a)$$

$$\begin{array}{|c|c|} \hline T & T \\ \hline \end{array} \otimes \begin{array}{|c|c|} \hline T & T \\ \hline \end{array} + 2M \begin{array}{|c|} \hline L \\ \hline \end{array} \otimes \begin{array}{|c|c|} \hline T & T \\ \hline \end{array} = 0, \quad (7.26b)$$

$$\begin{array}{|c|c|} \hline T & T \\ \hline \end{array} \otimes \begin{array}{|c|} \hline L \\ \hline \end{array} + 2M \begin{array}{|c|} \hline L \\ \hline \end{array} \otimes \begin{array}{|c|} \hline L \\ \hline \end{array} = 0, \quad (7.27)$$

$$\begin{array}{|c|c|} \hline L & L \\ \hline \end{array} \otimes \begin{array}{|c|c|} \hline T & T \\ \hline \end{array} + 2M \begin{array}{|c|c|} \hline L & L \\ \hline \end{array} \otimes \begin{array}{|c|} \hline L \\ \hline \end{array} = 0, \quad (7.28a)$$

$$\begin{array}{|c|c|} \hline T & T \\ \hline \end{array} \otimes \begin{array}{|c|c|} \hline T & T \\ \hline \end{array} + 2M \begin{array}{|c|c|} \hline T & T \\ \hline \end{array} \otimes \begin{array}{|c|} \hline L \\ \hline \end{array} = 0, \quad (7.28b)$$

$$\begin{array}{|c|} \hline L \\ \hline \end{array} \otimes \begin{array}{|c|c|} \hline T & T \\ \hline \end{array} + 2M \begin{array}{|c|} \hline L \\ \hline \end{array} \otimes \begin{array}{|c|} \hline L \\ \hline \end{array} = 0. \quad (7.29)$$

Since the transverse component of the highest spin state $\alpha_{-1}^T \cdots \alpha_{-1}^T \otimes \tilde{\alpha}_{-1}^T \cdots \tilde{\alpha}_{-1}^T$ at each fixed mass level gives the leading order scattering amplitude, there should have even number of T at each fixed mass level. Thus Eqs.(7.22c), (7.23b), (7.24c) and (7.25b) are subleading order in energy and are therefore irrelevant. Set $\begin{array}{|c|c|} \hline T & T \\ \hline \end{array} \otimes \begin{array}{|c|c|} \hline T & T \\ \hline \end{array} = 1$, we can solve the ratios from the remaining equations. The final result is

$\epsilon_{TT,TT}$	1
$\epsilon_{TT,LL} = \epsilon_{LL,TT}$	$1/(2M^2)$
$\epsilon_{LL,LL}$	$1/(4M^4)$
$\epsilon_{TT,L} = \epsilon_{L,TT}$	$-1/(2M)$
$\epsilon_{LL,L} = \epsilon_{L,LL}$	$-1/(4M^3)$
$\epsilon_{L,L}$	$1/(4M^2)$

which is exactly the tensor product of two pieces of open string ratios. This result is consistent with Eq.(7.16) calculated from the decoupling of high energy ZNS in the previous section.

C. Saddle-point calculation

In this section, we calculate the tree-level high energy closed string scattering amplitudes for arbitrary mass levels. We first review the calculation of high energy open string scattering amplitude. The (s, t) channel scattering amplitude with $V_2 = \alpha_{-1}^{\mu_1} \alpha_{-1}^{\mu_2} \dots \alpha_{-1}^{\mu_n} | 0, k \rangle$, the highest spin state at mass level $M^2 = 2(N-1)$, and three tachyons $V_{1,3,4}$ is [29]

$$\mathcal{T}_{N;st}^{\mu_1 \mu_2 \dots \mu_n} = \sum_{l=0}^N (-)^l \binom{N}{l} B\left(-\frac{s}{2} - 1 + l, -\frac{t}{2} - 1 + N - l\right) k_1^{(\mu_1)} \dots k_1^{\mu_{n-l}} k_3^{\mu_{n-l+1}} \dots k_3^{\mu_n}, \quad (7.30)$$

where $B(u, v) = \int_0^1 dx x^{u-1} (1-x)^{v-1}$ is the Euler beta function. It is now easy to calculate the general high energy scattering amplitude at the $M^2 = 2(N-1)$ level

$$\mathcal{T}_n^{TTT\dots}(s, t) \simeq [-2E^3 \sin \phi_{c.m.}]^N \mathcal{T}_N(s, t) \quad (7.31)$$

where $\mathcal{T}_N(s, t)$ is the high energy limit of $\frac{\Gamma(-\frac{s}{2}-1)\Gamma(-\frac{t}{2}-1)}{\Gamma(\frac{u}{2}+2)}$ with $s + t + u = 2N - 8$, and was previously [26, 27, 29] miscalculated to be

$$\begin{aligned} \tilde{\mathcal{T}}_{N;st} &\simeq \sqrt{\pi} (-1)^{N-1} 2^{-N} E^{-1-2N} \left(\sin \frac{\phi_{c.m.}}{2} \right)^{-3} \left(\cos \frac{\phi_{c.m.}}{2} \right)^{5-2N} \\ &\times \exp \left[-\frac{s \ln s + t \ln t - (s+t) \ln(s+t)}{2} \right] \end{aligned} \quad (7.32)$$

One can now generalize this result to multi-tensors. The (s, t) channel of open string high energy scattering amplitude at mass level (N_1, N_2, N_3, N_4) was calculated to be [26, 27, 29]

$$\mathcal{T}_{N_1 N_2 N_3 N_4; st}^{T^1 \dots T^2 \dots T^3 \dots T^4 \dots} = [-2E^3 \sin \phi_{c.m.}]^{\sum N_i} \mathcal{T}_{\sum N_i}(s, t). \quad (7.33)$$

In the above calculations, the scattering angle $\phi_{c.m.}$ in the center of mass frame is defined to be the angle between \vec{k}_1 and \vec{k}_3 . $s = -(k_1 + k_2)^2$, $t = -(k_2 + k_3)^2$ and $u = -(k_1 + k_3)^2$ are the Mandelstam variables. $M_i^2 = 2(N_i - 1)$ with N_i the mass level of the i th vertex. T^i in Eq.(7.33) is the transverse polarization of the i th vertex defined in Eq.(8). All other 4-point functions at mass level (N_1, N_2, N_3, N_4) were shown to be proportional to Eq.(7.33).

The corresponding (t, u) channel scattering amplitudes of Eqs.(7.31) and (7.33) can be obtained by replacing (s, t) in Eq.(7.32) by (t, u)

$$\begin{aligned} \mathcal{T}_N(t, u) &\simeq \sqrt{\pi}(-1)^{N-1}2^{-N}E^{-1-2N} \left(\sin \frac{\phi_{c.m.}}{2}\right)^{-3} \left(\cos \frac{\phi_{c.m.}}{2}\right)^{5-2N} \\ &\times \exp \left[-\frac{t \ln t + u \ln u - (t+u) \ln(t+u)}{2} \right]. \end{aligned} \quad (7.34)$$

We now claim that only (t, u) channel of the amplitude, Eq.(7.34), is suitable for saddle-point calculation. The previous saddle-point calculation for the (s, t) channel amplitude, Eq.(7.32), in the high energy expansion is misleading. The corrected high energy calculation of the (s, t) channel amplitude will be given in Eq.(7.51). The reason is as following. When calculating Eq.(7.31) from Eq.(7.30), one calculates the high energy limit of

$$\frac{\Gamma(-\frac{s}{2}-1)\Gamma(-\frac{t}{2}-1)}{\Gamma(\frac{u}{2}+2)}, s+t+u=2N-8, \quad (7.35)$$

in Eq.(7.30) by expanding the Γ function with the Stirling formula

$$\Gamma(x) \sim \sqrt{2\pi}x^{x-1/2}e^{-x}. \quad (7.36)$$

However, the above expansion is not suitable for negative real x as there are poles for $\Gamma(x)$ at $x = -N$, negative integers. Unfortunately, our high energy limit

$$s \sim 4E^2 \gg 0, \quad (7.37a)$$

$$t \sim -4E^2 \sin^2 \left(\frac{\phi_{c.m.}}{2} \right) \ll 0, \quad (7.37b)$$

$$u \sim -4E^2 \cos^2 \left(\frac{\phi_{c.m.}}{2} \right) \ll 0, \quad (7.37c)$$

contains this dangerous situation in the (s, t) channel calculation of Eq.(7.32). On the other hand, the corresponding high energy expansion of (t, u) channel scattering amplitude in Eq.(7.34) is well defined. Another evidence for this point is the following. When one uses the saddle point method to calculate the high energy open string scattering amplitudes in the (s, t) channel, the saddle-point we identified was [29–31]

$$x_0 = \frac{s}{s+t} = \frac{1}{1 - \sin^2(\phi/2)} > 1, \quad (7.38)$$

which is out of the integration range $(0, 1)$. Therefore, we can not trust the saddle point calculation for the (s, t) channel scattering amplitude. On the other hand, the corresponding

saddle-point calculation for the (t, u) channel scattering amplitude is safe since the saddle-point x_0 is within the integration range $(1, \infty)$. This subtle situation becomes crucial and relevant when one tries to calculate the high energy closed string scatterings amplitude and compare them with the open string ones.

We now discuss the high energy closed string scattering amplitudes. There exists a celebrated formula by Kawai, Lewellen and Tye (KLT), which expresses the relation between tree amplitudes of closed and open string ($\alpha'_{\text{closed}} = 4\alpha'_{\text{open}} = 2$)

$$A_{\text{closed}}^{(4)}(s, t, u) = \sin(\pi k_2 \cdot k_3) A_{\text{open}}^{(4)}(s, t) \bar{A}_{\text{open}}^{(4)}(t, u) \quad (7.39)$$

To calculate the high energy closed string scattering amplitudes, one encounters the difficulty of calculation of high energy open string amplitude in the (s, t) channel discussed above. To avoid this difficulty, we can use the well known formula

$$\Gamma(x) = \frac{\pi}{\sin(\pi x) \Gamma(1-x)} \quad (7.40)$$

to calculate the large negative x expansion of the Γ function. We first discuss the high energy four-tachyon scattering amplitude which already existed in the literature. We can express the open string (s, t) channel amplitude in terms of the (t, u) channel amplitude,

$$\begin{aligned} A_{\text{open}}^{(4\text{-tachyon})}(s, t) &= \frac{\Gamma(-\frac{s}{2} - 1) \Gamma(-\frac{t}{2} - 1)}{\Gamma(\frac{u}{2} + 2)} \\ &= \frac{\sin(\pi u/2) \Gamma(-\frac{t}{2} - 1) \Gamma(-\frac{u}{2} - 1)}{\sin(\pi s/2) \Gamma(\frac{s}{2} + 2)} \\ &\equiv \frac{\sin(\pi u/2)}{\sin(\pi s/2)} A_{\text{open}}^{(4\text{-tachyon})}(t, u), \end{aligned} \quad (7.41)$$

which we know how to calculate the high energy limit. Note that for the four-tachyon case, $\bar{A}_{\text{open}}^{(4)}(t, u) = A_{\text{open}}^{(4)}(t, u)$ in Eq.(7.39). The KLT formula, Eq.(7.39), can then be used to express the closed string four-tachyon scattering amplitude in terms of that of open string in the (t, u) channel

$$A_{\text{closed}}^{(4\text{-tachyon})}(s, t, u) = \frac{\sin(\pi t/2) \sin(\pi u/2)}{\sin(\pi s/2)} A_{\text{open}}^{(4\text{-tachyon})}(t, u) A_{\text{open}}^{(4\text{-tachyon})}(t, u). \quad (7.42)$$

The high energy limit of open string four-tachyon amplitude in the (t, u) channel can be easily calculated to be

$$A_{\text{open}}^{(4\text{-tachyon})}(t, u) \simeq (stu)^{-\frac{3}{2}} \exp\left(-\frac{s \ln s + t \ln t + u \ln u}{2}\right), \quad (7.43)$$

which gives the corresponding amplitude in the (s, t) channel

$$A_{\text{open}}^{(4\text{-tachyon})}(s, t) \simeq \frac{\sin(\pi u/2)}{\sin(\pi s/2)} (stu)^{-\frac{3}{2}} \exp\left(-\frac{s \ln s + t \ln t + u \ln u}{2}\right) \quad (7.44)$$

The high energy limit of closed string four-tachyon scattering amplitude can then be calculated, through the KLT formula, to be

$$A_{\text{closed}}^{(4\text{-tachyon})}(s, t, u) \simeq \frac{\sin(\pi t/2) \sin(\pi u/2)}{\sin(\pi s/2)} (stu)^{-3} \exp\left(-\frac{s \ln s + t \ln t + u \ln u}{4}\right) \quad (7.45)$$

The exponential factor in Eq.(7.43) was first discussed by Veneziano [36]. Our result for the high energy closed string four-tachyon amplitude in Eq.(7.45) differs from the one calculated in the literature [1, 2] by an oscillating factor $\frac{\sin(\pi t/2) \sin(\pi u/2)}{\sin(\pi s/2)}$. We stress here that our results for Eqs.(7.43), (7.44) and (7.45) are consistent with the KLT formula, while the previous calculation in [1, 2] is NOT.

One might try to use the saddle-point method to calculate the high energy closed string scattering amplitude. The closed string four-tachyon scattering amplitude is

$$\begin{aligned} A_{\text{closed}}^{(4\text{-tachyon})}(s, t, u) &= \int dx dy \exp\left(\frac{k_1 \cdot k_2}{2} \ln|z| + \frac{k_2 \cdot k_3}{2} \ln|1-z|\right) \\ &= \int dx dy (x^2 + y^2)^{-2} [(1-x)^2 + y^2]^{-2} \\ &\quad \cdot \exp\left\{-\frac{s}{8} \ln(x^2 + y^2) - \frac{t}{8} \ln[(1-x)^2 + y^2]\right\} \\ &\equiv \int dx dy (x^2 + y^2)^{-2} [(1-x)^2 + y^2]^{-2} \exp[-K f(x, y)] \end{aligned} \quad (7.46)$$

where $K = \frac{s}{8}$ and $f(x, y) = \ln(x^2 + y^2) - \tau \ln[(1-x)^2 + y^2]$ with $\tau = -\frac{t}{s}$. One can then calculate the "saddle-point" of $f(x, y)$ to be

$$\nabla f(x, y) \big|_{x_0=\frac{1}{1-\tau}, y_0=0} = 0. \quad (7.47)$$

The high energy limit of the closed string four-tachyon scattering amplitude is then calculated to be

$$A_{\text{closed}}^{(4\text{-tachyon})}(s, t, u) \simeq \frac{2\pi}{K \sqrt{\det \frac{\partial^2 f(x_0, y_0)}{\partial x \partial y}}} \exp[-K f(x_0, y_0)] \simeq (stu)^{-3} \exp\left(-\frac{s \ln s + t \ln t + u \ln u}{4}\right), \quad (7.48)$$

which is consistent with the previous one calculated in the literature [1, 2], but is different from our result in Eq.(7.45). However, one notes that

$$\frac{\partial^2 f(x_0, y_0)}{\partial x^2} = \frac{2(1-\tau)^3}{\tau} = -\frac{\partial^2 f(x_0, y_0)}{\partial y^2}, \quad \frac{\partial^2 f(x_0, y_0)}{\partial x \partial y} = 0, \quad (7.49)$$

which means that (x_0, y_0) is NOT the local minimum of $f(x, y)$, and one should not trust this saddle-point calculation. This is the third evidence to see that there is no clear definition of saddle-point in the calculation of the high energy open string scattering amplitude in the (s, t) channel, and thus the invalid saddle-point calculation of high energy closed string scattering amplitude.

Finally we calculate the high energy closed string scattering amplitudes for arbitrary mass levels. The (t, u) channel open string scattering amplitude with $V_2 = \alpha_{-1}^{\mu_1} \alpha_{-1}^{\mu_2} \dots \alpha_{-1}^{\mu_n} | 0, k \rangle$, the highest spin state at mass level $M^2 = 2(N-1)$, and three tachyons $V_{1,3,4}$ can be calculated to be

$$\mathcal{T}_{N;tu}^{\mu_1 \mu_2 \dots \mu_n} = \sum_{l=0}^N \binom{N}{l} B \left(-\frac{t}{2} + N - l - 1, -\frac{u}{2} - 1 \right) k_1^{(\mu_1)} \dots k_1^{\mu_{N-l}} k_3^{\mu_{N-l+1}} k_3^{\mu_N}. \quad (7.50)$$

In calculating Eq.(7.50), we have used the Mobius transformation $y = \frac{x-1}{x}$ to change the integration region from $(1, \infty)$ to $(0, 1)$. One notes that Eq.(7.50) is NOT the same as Eq.(7.30) with (s, t) replaced by (t, u) , as one would have expected from the four-tachyon case discussed in the paragraph after Eq.(7.39). In the high energy limit, one easily sees that

$$\mathcal{T}_N(s, t) \simeq (-)^N \frac{\sin(\pi u/2)}{\sin(\pi s/2)} \mathcal{T}_N(t, u), \quad (7.51)$$

which is the generalization of Eq.(7.41) to arbitrary mass levels. Eq.(7.51) is part of the string BCJ relations which will be discussed in the next section. Eq.(7.51) is the correction of Eqs.(7.31) and (7.32) as claimed in the paragraph after Eq.(7.34). The (s, t) channel of high energy open string scattering amplitudes at mass level (n_1, n_2, n_3, n_4) can then be written as, apart from an overall constant,

$$\begin{aligned} A_{\text{open}}^{(4)}(s, t) &\simeq (-)^{\Sigma N_i} \frac{\sin(\pi u/2)}{\sin(\pi s/2)} [-2E^3 \sin \phi_{c.m.}]^{\Sigma N_i} \mathcal{T}_{\Sigma N_i}(t, u) \\ &\simeq (-)^{\Sigma N_i} \frac{\sin(\pi u/2)}{\sin(\pi s/2)} (stu)^{\frac{\Sigma N_i - 3}{2}} \exp \left(-\frac{s \ln s + t \ln t + u \ln u}{2} \right). \end{aligned} \quad (7.52)$$

Finally the total high energy open string scattering amplitude is the sum of (s, t) , (t, u) and (u, s) channel amplitudes, and can be calculated to be

$$A_{\text{open}}^{(4)} \simeq (-)^{\Sigma N_i} \frac{\sin(\pi s/2) + \sin(\pi t/2) + \sin(\pi u/2)}{\sin(\pi s/2)} (stu)^{\frac{\Sigma N_i - 3}{2}} \exp \left(-\frac{s \ln s + t \ln t + u \ln u}{2} \right). \quad (7.53)$$

By using Eqs.(7.39) and (7.51), the high energy closed string scattering amplitude at mass level (N_1, N_2, N_3, N_4) is calculated to be, apart from an overall constant,

$$\begin{aligned} A_{\text{closed}}^{(4)}(s, t, u) &\simeq (-)^{\Sigma N_i} \frac{\sin(\pi t/2) \sin(\pi u/2)}{\sin(\pi s/2)} [-2E^3 \sin \phi_{c.m.}]^{2\Sigma N_i} \mathcal{T}_{\Sigma N_i}(t, u)^2 \\ &\simeq (-)^{\Sigma N_i} \frac{\sin(\pi t/2) \sin(\pi u/2)}{\sin(\pi s/2)} (stu)^{\Sigma N_i - 3} \exp\left(-\frac{s \ln s + t \ln t + u \ln u}{4}\right), \end{aligned} \quad (7.54)$$

where $\mathcal{T}_{\Sigma N_i}(t, u)$ is given by Eq.(7.34). For the case of four-tachyon scattering amplitude at mass level $(0, 0, 0, 0)$, Eq.(7.54) reduces to Eq.(7.45). All other high energy closed string scattering amplitudes at mass level (N_1, N_2, N_3, N_4) are proportional to Eq.(7.54). The proportionality constants are the tensor product of two pieces of open string ratios.

D. String BCJ relations

In 2008, the four point BCJ relations [37, 40, 123–125] for Yang-Mills gluon color-stripped scattering amplitudes A were pointed out and calculated to be

$$\begin{aligned} tA(k_1, k_4, k_2, k_3) - sA(k_1, k_3, k_4, k_2) &= 0, \\ sA(k_1, k_2, k_3, k_4) - uA(k_1, k_4, k_2, k_3) &= 0, \\ uA(k_1, k_3, k_4, k_2) - tA(k_1, k_2, k_3, k_4) &= 0, \end{aligned} \quad (7.55)$$

which relates field theory scattering amplitudes in the s , t and u channels. In the following, we will discuss the relation for s and u channel amplitudes only. Other relations can be similarly discussed.

For string theory, in contrast to the field theory BCJ relations, one has to deal with scattering amplitudes of infinite number of string states. For the tachyon state, the string BCJ relation was first calculated in 2006 to be Eq.(7.41) [35] which can be rewritten as

$$A_{\text{open}}^{(4\text{-tachyon})}(s, t) \equiv \frac{\sin(\pi k_2 \cdot k_4)}{\sin(\pi k_1 \cdot k_2)} A_{\text{open}}^{(4\text{-tachyon})}(t, u). \quad (7.56)$$

This relation for tachyon is valid for all energies. For *all* other higher spin string states at arbitrary mass levels, the high energy limit of string BCJ relation was worked out to be Eq.(7.51) [35] and can be rewritten as

$$\mathcal{T}_N(s, t) \simeq \frac{\sin(\pi k_2 \cdot k_4)}{\sin(\pi k_1 \cdot k_2)} \mathcal{T}_N(t, u). \quad (7.57)$$

Note that unlike the case of tachyon in Eq.(7.56), this relation was proved only for high energy limit. The result of Eq.(7.57) was based on two calculations. The first calculation was done for amplitudes in Eq.(7.50) and Eq.(7.30). Although the calculations in Eq.(7.50) and Eq.(7.30) were done only for three tachyons and one leading Regge trajectory higher spin state in the second vertex, it can be easily extended to three arbitrary string states and one leading Regge trajectory higher spin state in the high energy limit, and Eq.(7.57) is still valid. The second calculation was based on the fact that high energy, fixed angle amplitudes for states differ from leading Regge trajectory higher spin state in the second vertex are all proportional to each other at each fixed mass level as were shown in Eq.(5.60).

The two relations in Eq.(7.56) and Eq.(7.57) can be written as the four point *string BCJ relation* which are valid to all energies as

$$A_{\text{open}}^{(4)}(s, t) = \frac{\sin(\pi k_2 \cdot k_4)}{\sin(\pi k_1 \cdot k_2)} \bar{A}_{\text{open}}^{(4)}(t, u) \quad (7.58)$$

if one can generalize the proof of Eq.(7.51) to all energies. This was done in a paper based on monodromy of integration for string amplitudes published in 2009 [37]. The motivation for the author in [37] to calculate Eq.(7.58) was different from the calculation done in Eq.(7.57). It was based on the field theory BCJ relation [40]. An explicit proof of Eq.(7.58) for arbitrary four string states and all kinematic regimes was given very recently in [38, 39].

Note that for the supersymmetric case, there is no tachyon and the low energy massless limit of Eq.(7.58) reproduces the second equation of Eq.(7.55). Recently the mass level dependent of Eq.(7.58) was calculated to be [38, 39]

$$\frac{A_{st}^{(p,r,q)}}{A_{tu}^{(p,r,q)}} = (-1)^N \frac{B(-M_1 M_2 + 1, \frac{M_1 M_2}{2})}{B(\frac{M_1 M_2}{2}, \frac{M_1 M_2}{2})} \simeq \frac{\sin \pi(k_2 \cdot k_4)}{\sin \pi(k_1 \cdot k_2)}$$

by taking the *nonrelativistic* limit $|\vec{k}_2| \ll M_S$ of Eq.(7.58). In Eq.(0.52), B was the beta function, and k_1 , k_3 and k_4 were taken to be tachyons, and k_2 was the following tensor string state

$$V_2 = (i\partial X^T)^p (i\partial X^L)^r (i\partial X^P)^q e^{ik_2 X} \quad (7.59)$$

where

$$N = p + r + q, \quad M_2^2 = 2(N - 1). \quad (7.60)$$

The generalization of the four point function relation in Eq.(7.58) to higher point string amplitudes can be found in [37]. It is interesting to see that historically the four point (high

energy) string BCJ relations Eq.(7.41) and Eq.(7.51) [35] were discovered even earlier than the field theory BCJ relations Eq.(7.55)! [40].

In conclusion of this chapter, we have used the methods of decoupling of high energy ZNS and the high energy Virasoro constraints to calculate the ratios among high energy closed string scattering amplitudes of different string states. The result is exactly the tensor product of two pieces of open string ratios calculated before. However, we clarify the previous saddle-point calculation for high energy open string scattering amplitudes and show that only (t, u) channel of the amplitudes is suitable for saddle-point calculation. We also discuss three evidences, Eq.(7.37c), Eq.(7.38) and Eq.(7.49), to show that saddle-point calculation for high energy closed string scattering amplitudes is not reliable. Instead of using saddle-point calculation adopted before, we then propose to use the formula of Kawai, Lewellen and Tye (KLT) to calculate the high energy closed string scattering amplitudes for *arbitrary* mass levels.

For the case of high energy closed string four-tachyon amplitude, our result differs from the previous one of Gross and Mende, which is NOT consistent with KLT formula, by an oscillating factor. The oscillating prefactors in Eqs.(7.53) and (7.54) imply the existence of infinitely many zeros and poles in the string scattering amplitudes even in the high energy limit. Physically, the presence of poles simply reflects the fact that there are infinite number of resonances in the string spectrum [9], and the presence of zeros reflects the coherence of string scattering. In addition, the oscillating prefactors are crucial to discuss the string BCJ relations.

VIII. HARD SUPERSTRING SCATTERINGS

In this chapter, we consider high energy scattering amplitudes for the NS sector of $10D$ open superstring theory [33]. Based on the calculations of $26D$ bosonic open string [30, 31, 44], all the three independent calculations of bosonic string, namely the decoupling of high energy ZNS (HZNS), the Virasoro constraints and the saddle-point calculation can be generalized to scattering amplitudes of string states with polarizations on the scattering plane of superstring. All three methods give the consistent results [33].

In addition, we discover new leading order high energy scattering amplitudes, which are still proportional to the previous ones, with polarizations *orthogonal* to the scattering plane

[33]. These scattering amplitudes are of subleading order in energy for the case of $26D$ open bosonic string theory. The existence of these new high energy scattering amplitudes is due to the worldsheet fermion exchange in the correlation functions and is, presumably, related to the high energy massive fermionic scattering amplitudes in the R-sector of the theory. We thus conjecture that the validity of Gross's two conjectures on high energy stringy symmetry persists for superstring theory.

A. Decoupling of high energy ZNS

We will first consider high energy scattering amplitudes of string states with polarizations on the scattering plane. Those with polarizations orthogonal to the scattering plane will be discussed in section VIII.D. It can be argued that there are four types of high energy scattering amplitudes for states in the NS sector with even GSO parity [33]

$$|n, 2m, q\rangle \otimes \left| b_{-\frac{1}{2}}^T \right\rangle \equiv (\alpha_{-1}^T)^{n-2m-2q} (\alpha_{-1}^L)^{2m} (\alpha_{-2}^L)^q (b_{-\frac{1}{2}}^T) |0, k\rangle, \quad (8.1)$$

$$|n, 2m+1, q\rangle \otimes \left| b_{-\frac{1}{2}}^L \right\rangle \equiv (\alpha_{-1}^T)^{n-2m-2q-1} (\alpha_{-1}^L)^{2m+1} (\alpha_{-2}^L)^q (b_{-\frac{1}{2}}^L) |0, k\rangle, \quad (8.2)$$

$$|n, 2m, q\rangle \otimes \left| b_{-\frac{3}{2}}^L \right\rangle \equiv (\alpha_{-1}^T)^{n-2m-2q} (\alpha_{-1}^L)^{2m} (\alpha_{-2}^L)^q (b_{-\frac{3}{2}}^L) |0, k\rangle, \quad (8.3)$$

$$|n, 2m, q\rangle \otimes \left| b_{-\frac{1}{2}}^T b_{-\frac{1}{2}}^L b_{-\frac{3}{2}}^L \right\rangle \equiv (\alpha_{-1}^T)^{n-2m-2q} (\alpha_{-1}^L)^{2m} (\alpha_{-2}^L)^q (b_{-\frac{1}{2}}^T) (b_{-\frac{1}{2}}^L) (b_{-\frac{3}{2}}^L) |0, k\rangle \quad (8.4)$$

Note that the number of α_{-1}^L operator in Eq.(8.2) is odd. In the OCFQ spectrum of open superstring, the solutions of physical states conditions include positive-norm propagating states and two types of ZNS. In the NS sector, the latter are [9]

$$\text{Type I : } G_{-\frac{1}{2}} |x\rangle, \text{ where } G_{\frac{1}{2}} |x\rangle = G_{\frac{3}{2}} |x\rangle = 0, \quad L_0 |x\rangle = 0; \quad (8.5)$$

$$\text{Type II : } (G_{-\frac{3}{2}} + 2G_{-\frac{1}{2}} L_{-1}) |\tilde{x}\rangle, \text{ where } G_{\frac{1}{2}} |\tilde{x}\rangle = G_{\frac{3}{2}} |\tilde{x}\rangle = 0, \quad (L_0 + 1) |\tilde{x}\rangle = 0. \quad (8.6)$$

While Type I states have zero-norm at any space-time dimension, Type II states have zero-norm *only* at $D = 10$. We will show that, for each fixed mass level, all high energy scattering amplitudes corresponding to states in Eqs.(8.1)-(8.4) are proportional to each other, and the proportionality constants can be determined from the decoupling of two types of ZNS, Eqs.(8.5) and (8.6) in the high energy limit. For simplicity, based on the result of Eq.(5.60),

one needs only calculate the proportionality constants among the scattering amplitudes of the following four lower mass level states

$$|2, 0, 0\rangle \otimes \left| b_{-\frac{1}{2}}^T \right\rangle \equiv (\alpha_{-1}^T)^2 (b_{-\frac{1}{2}}^T) |0, k\rangle, \quad (8.7)$$

$$|2, 1, 0\rangle \otimes \left| b_{-\frac{1}{2}}^L \right\rangle \equiv (\alpha_{-1}^T)(\alpha_{-1}^L)(b_{-\frac{1}{2}}^L) |0, k\rangle, \quad (8.8)$$

$$|1, 0, 0\rangle \otimes \left| b_{-\frac{3}{2}}^L \right\rangle \equiv (\alpha_{-1}^T)(b_{-\frac{3}{2}}^L) |0, k\rangle, \quad (8.9)$$

$$|0, 0, 0\rangle \otimes \left| b_{-\frac{1}{2}}^T b_{-\frac{1}{2}}^L b_{-\frac{3}{2}}^L \right\rangle \equiv (b_{-\frac{1}{2}}^T)(b_{-\frac{1}{2}}^L)(b_{-\frac{3}{2}}^L) |0, k\rangle \quad (8.10)$$

Other proportionality constants for higher mass level can be obtained through Eqs.(5.60) and (8.7)-(8.10). To calculate the ratio among the high energy scattering amplitudes corresponding to states in Eqs.(8.8) and (8.9), we use the decoupling of the Type I HZNS at mass level $M^2 = 2$

$$G_{-\frac{1}{2}}(\alpha_{-1}^L) |0, k\rangle = [M(\alpha_{-1}^L)(b_{-\frac{1}{2}}^L) + (b_{-\frac{1}{2}}^L)] |0, k\rangle. \quad (8.11)$$

Eq.(8.11) gives the ratio for states at mass level $M^2 = 4$

$$(\alpha_{-1}^T)(b_{-\frac{3}{2}}^L) |0, k\rangle : (\alpha_{-1}^T)(\alpha_{-1}^L)(b_{-\frac{1}{2}}^L) |0, k\rangle = M : -1. \quad (8.12)$$

We have used an abbreviated notation for the scattering amplitudes on the l.h.s. of Eq.(8.12).

The HZNS in Eq.(8.11) is the high energy limit of the vector ZNS at mass level $M^2 = 2$

$$G_{-\frac{1}{2}} |x\rangle = [k_{(\mu} \theta_{\nu)} \alpha_{-1}^\mu b_{-\frac{1}{2}}^\nu + \theta \cdot b_{-\frac{3}{2}}] |0, k\rangle, \quad (8.13)$$

where

$$|x\rangle = [\theta \cdot \alpha_{-1} + \frac{1}{2} k \cdot b_{-\frac{1}{2}} \theta \cdot b_{-\frac{1}{2}}] |0, k\rangle, k \cdot \theta = 0 \quad (8.14)$$

In fact, in the high energy limit, $\theta = e^L$, so $|x\rangle \rightarrow (\alpha_{-1}^L) |0, k\rangle$ and Eq.(8.13) reduces to Eq.(8.11). To calculate the ratio among the high energy scattering amplitudes corresponding to states in Eqs.(8.7) and (8.9), we use the decoupling of the Type II HZNS at mass level $M^2 = 4$

$$G_{-\frac{3}{2}}(\alpha_{-1}^T) |0, k\rangle = [M(\alpha_{-1}^T)(b_{-\frac{3}{2}}^L) + (\alpha_{-1}^T)^2 (b_{-\frac{1}{2}}^T)] |0, k\rangle. \quad (8.15)$$

Eq.(8.15) gives the ratio

$$(\alpha_{-1}^T)(b_{-\frac{3}{2}}^L) |0, k\rangle : (\alpha_{-1}^T)^2 (b_{-\frac{1}{2}}^T) |0, k\rangle = 1 : -M. \quad (8.16)$$

Finally, To calculate the ratio among the high energy scattering amplitudes corresponding to states in Eqs.(8.7) and (8.10), we use the decoupling of the Type II HZNS at mass level $M^2 = 4$

$$G_{-\frac{3}{2}}(b_{-\frac{1}{2}}^T)(b_{-\frac{1}{2}}^L) |0, k\rangle \equiv [M(b_{-\frac{1}{2}}^T)(b_{-\frac{1}{2}}^L)(b_{-\frac{3}{2}}^L) + (\alpha_{-2}^L)(b_{-\frac{1}{2}}^T)] |0, k\rangle. \quad (8.17)$$

Eq.(8.17) gives the ratio

$$(b_{-\frac{1}{2}}^T)(b_{-\frac{1}{2}}^L)(b_{-\frac{3}{2}}^L) |0, k\rangle : (\alpha_{-2}^L)(b_{-\frac{1}{2}}^T) |0, k\rangle = 1 : -M. \quad (8.18)$$

On the other hand, Eq.(5.94) gives

$$(\alpha_{-2}^L)(b_{-\frac{1}{2}}^T) |0, k\rangle : (\alpha_{-1}^T)^2(b_{-\frac{1}{2}}^T) |0, k\rangle = 1 : -2M. \quad (8.19)$$

We conclude that

$$(b_{-\frac{1}{2}}^T)(b_{-\frac{1}{2}}^L)(b_{-\frac{3}{2}}^L) |0, k\rangle : (\alpha_{-1}^T)^2(b_{-\frac{1}{2}}^T) |0, k\rangle = 1 : 2M^2. \quad (8.20)$$

Eqs.(8.12),(8.16) and (8.20) give the proportionality constants among high energy scattering amplitudes corresponding to states in Eqs.(8.7)-(8.10). Finally, by using Eq.(5.60), one can then easily calculate the proportionality constants among high energy scattering amplitudes corresponding to states in Eqs.(8.1)-(8.4). The results will be presented in Eqs.(12.1)-(12.4) of chapter XII

B. Virasoro constraints

In this section, we will use the method of Virasoro constraints to derive the ratios between the physical states in the NS sector. In the superstring theory, the physical state $|\phi\rangle$ in the NS sector should satisfy the following conditions:

$$\left(L_0 - \frac{1}{2}\right) |\phi\rangle = 0, \quad (8.21)$$

$$L_m |\phi\rangle = 0, \quad m = 1, 2, 3, \dots, \quad (8.22)$$

$$G_r |\phi\rangle = 0, \quad r = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots, \quad (8.23)$$

where the L_m and G_r are super Virasoro operators in the NS sector,

$$L_m = \frac{1}{2} \sum_n : \alpha_{m-n} \cdot \alpha_n : + \frac{1}{4} \sum_r (2r - m) : \psi_{m-r} \cdot \psi_r :, \quad (8.24)$$

$$G_r = \sum_n \alpha_n \cdot \psi_{r-n}. \quad (8.25)$$

These super Virasoro operators satisfy the following superconformal algebra,

$$\begin{aligned}
[L_m, L_n] &= (m - n) L_{m+n} + \frac{1}{8} D (m^3 - m) \delta_{m+n}, \\
[L_m, G_r] &= \left(\frac{1}{2} m - r \right) G_{m+r}, \\
\{G_r, G_s\} &= 2L_{r+s} + \frac{1}{2} D \left(r^2 - \frac{1}{4} \right) \delta_{r+s}.
\end{aligned} \tag{8.26}$$

Using the above superconformal algebra, the Virasoro conditions (8.22) and (8.23) reduce to the following simple form,

$$G_{1/2} |\phi\rangle = 0, \tag{8.27}$$

$$G_{3/2} |\phi\rangle = 0. \tag{8.28}$$

In the following, we will use the reduced Virasoro conditions (8.27) and (8.28) to determine the ratios between the physical states in the NS sector in the high energy limit.

To warm up, let us consider the mass level at $M^2 = 2$ first. The most general state in the NS sector at this mass level can be written as

$$|2\rangle = \left\{ \begin{array}{c} \boxed{\mu} \psi_{-\frac{3}{2}}^\mu + \boxed{\mu} \otimes \boxed{\nu} \alpha_{-1}^\mu \psi_{-\frac{1}{2}}^\nu + \begin{array}{c} \boxed{\mu} \\ \boxed{\nu} \psi_{-\frac{1}{2}}^\mu \psi_{-\frac{1}{2}}^\nu \psi_{-\frac{1}{2}}^\sigma \\ \boxed{\sigma} \end{array} \end{array} \right\} |0\rangle_{NS}, \tag{8.29}$$

where we use the Young tableaux to represent the coefficients of different tensors. The properties of symmetry and anti-symmetry can be easily and clearly described in this representation.

We then apply the reduced Virasoro conditions (8.27) and (8.28) to the state (8.29). It is easy to obtain

$$G_{1/2} |2\rangle = \alpha_{-1}^\mu \left\{ \begin{array}{c} \boxed{\mu} + k^\nu \boxed{\mu} \otimes \boxed{\nu} \end{array} \right\} + \psi_{-\frac{1}{2}}^\mu \psi_{-\frac{1}{2}}^\nu \left\{ \begin{array}{c} \boxed{\mu} \otimes \boxed{\nu} - \boxed{\nu} \otimes \boxed{\mu} + 3k^\sigma \begin{array}{c} \boxed{\mu} \\ \boxed{\nu} \\ \boxed{\sigma} \end{array} \end{array} \right\}, \tag{8.30a}$$

$$G_{3/2} |2\rangle = \boxed{\mu} k^\mu + \boxed{\mu} \otimes \boxed{\nu} \eta^{\mu\nu}, \tag{8.30b}$$

which leads to the following equations,

$$\boxed{\mu} + k^\nu \boxed{\mu} \otimes \boxed{\nu} = 0, \quad (8.31a)$$

$$\boxed{\mu} \otimes \boxed{\nu} - \boxed{\nu} \otimes \boxed{\mu} + 3k^\sigma \begin{array}{c} \boxed{\mu} \\ \boxed{\nu} \\ \boxed{\sigma} \end{array} = 0, \quad (8.31b)$$

$$\boxed{\mu} k^\mu + \boxed{\mu} \otimes \boxed{\nu} \eta^{\mu\nu} = 0. \quad (8.31c)$$

To solve the above equation, we first take the high energy limit by letting $\mu \rightarrow (L, T)$ and

$$k^\mu \rightarrow M (e^L)^\mu, \quad \eta^{\mu\nu} \rightarrow (e^T)^\mu (e^T)^\nu. \quad (8.32)$$

The above equations reduce to

$$\boxed{\mu} + M \boxed{\mu} \otimes \boxed{L} = 0, \quad (8.33)$$

$$\boxed{\mu} \otimes \boxed{\nu} - \boxed{\nu} \otimes \boxed{\mu} = 0, \quad (8.34)$$

$$M \boxed{L} + \boxed{T} \otimes \boxed{T} = 0. \quad (8.35)$$

At this mass level, the terms with odd number of T 's will be sub-leading in the high energy limit and be ignored, the resulting equations contain only terms with even number of T 's as following,

$$\boxed{L} + M \boxed{L} \otimes \boxed{L} = 0, \quad (8.36)$$

$$M \boxed{L} + \boxed{T} \otimes \boxed{T} = 0. \quad (8.37)$$

The ratio of the coefficients then can be obtained as

ε_{TT}	$M^2 (= 2)$
ε_{LL}	1
ε_L	$-M (= -\sqrt{2})$

(8.38)

Now we will consider the general mass level at $M^2 = 2(N-1)$. At this mass level, the most general state can be written as

$$|N\rangle = \sum_{\{m_j, m_r\}} \left[\bigotimes_{j=1}^N \frac{1}{j^{m_j} m_j!} \boxed{\mu_1^j} \cdots \boxed{\mu_{m_j}^j} \alpha_{-j}^{\mu_1^j \cdots \mu_{m_j}^j} \bigotimes_{r=1/2}^{N-1/2} \frac{1}{m_r!} \boxed{\nu_1^r} \cdots \boxed{\nu_{m_r}^r} \right]^T \psi_{-r}^{\nu_1^r \cdots \nu_{m_r}^r} |0, k\rangle, \quad (8.39)$$

where

$$\begin{array}{|c|c|c|} \hline \nu_1^r & \cdots & \nu_{m_r}^r \\ \hline \end{array}^T = \begin{array}{|c|} \hline \nu_1^r \\ \hline \vdots \\ \hline \nu_{m_r}^r \\ \hline \end{array}, \quad (8.40)$$

and we have defined the abbreviation

$$\alpha_{-j}^{\mu_1^j \cdots \mu_{m_j}^j} \equiv \alpha_{-j}^{\mu_1^j} \cdots \alpha_{-j}^{\mu_{m_j}^j} \quad \text{and} \quad \psi_{-r}^{\nu_1^r \cdots \nu_{m_r}^r} \equiv \psi_{-r}^{\nu_1^r} \cdots \psi_{-r}^{\nu_{m_r}^r}, \quad (8.41)$$

with m_j (m_r) is the number of the operator α_{-j}^μ (ψ_{-r}^ν) for $j \in Z$ and $r \in Z + 1/2$. The summation runs over all possible m_j (m_r) with the constrain

$$\sum_{j=1}^N j m_j + \sum_{r=1/2}^{N-1/2} r m_r = N - \frac{1}{2} \quad \text{with} \quad m_j, m_r \geq 0, \quad (8.42)$$

so that the total mass square is $2(N - 1)$.

Solving the constraints (8.27) and (8.28) in the high energy limit, the ratios between the physical states in the NS sector are obtained as (see Appendix B for detail)

$$\begin{aligned} & \underbrace{\begin{array}{|c|c|c|} \hline T & \cdots & T \\ \hline \end{array}}_{N-2m_2-2k} \underbrace{\begin{array}{|c|c|c|} \hline L & \cdots & L \\ \hline \end{array}}_{2k} \otimes \underbrace{\begin{array}{|c|c|c|} \hline L & \cdots & L \\ \hline \end{array}}_{m_2} \otimes \boxed{0} \otimes \boxed{L} \\ &= \left(-\frac{1}{2M}\right)^{m_2} \left(-\frac{1}{2M}\right)^k \frac{(2k-1)!!}{(-M)^k} \underbrace{\begin{array}{|c|c|c|} \hline T & \cdots & T \\ \hline \end{array}}_N \otimes \boxed{0} \otimes \boxed{0} \otimes \boxed{L}, \end{aligned} \quad (8.43)$$

$$\begin{aligned} & \underbrace{\begin{array}{|c|c|c|} \hline T & \cdots & T \\ \hline \end{array}}_{N-2m_2-2k} \underbrace{\begin{array}{|c|c|c|} \hline L & \cdots & L \\ \hline \end{array}}_{2k+1} \otimes \underbrace{\begin{array}{|c|c|c|} \hline L & \cdots & L \\ \hline \end{array}}_{m_2} \otimes \boxed{L} \otimes \boxed{0} \\ &= \left(-\frac{1}{2M}\right)^{m_2} \left(-\frac{1}{2M}\right)^k \frac{(2k+1)!!}{(-M)^{k+1}} \underbrace{\begin{array}{|c|c|c|} \hline T & \cdots & T \\ \hline \end{array}}_N \otimes \boxed{0} \otimes \boxed{0} \otimes \boxed{L}, \end{aligned} \quad (8.44)$$

$$\begin{aligned} & \underbrace{\begin{array}{|c|c|c|} \hline T & \cdots & T \\ \hline \end{array}}_{N-2m_2-2k+1} \underbrace{\begin{array}{|c|c|c|} \hline L & \cdots & L \\ \hline \end{array}}_{2k} \otimes \underbrace{\begin{array}{|c|c|c|} \hline L & \cdots & L \\ \hline \end{array}}_{m_2} \otimes \boxed{T} \otimes \boxed{0} \\ &= \left(-\frac{1}{2M}\right)^{m_2} \left(-\frac{1}{2M}\right)^k \frac{(2k-1)!!}{(-M)^{k-1}} \underbrace{\begin{array}{|c|c|c|} \hline T & \cdots & T \\ \hline \end{array}}_N \otimes \boxed{0} \otimes \boxed{0} \otimes \boxed{L}, \end{aligned} \quad (8.45)$$

$$\begin{aligned}
& \underbrace{\boxed{T \cdots T}}_{N-2m_2-2k+1} \underbrace{\boxed{L \cdots L}}_{2k} \otimes \underbrace{\boxed{L \cdots L}}_{m_2-1} \otimes \boxed{T \cdots L}^T \otimes \boxed{L} \\
&= \left(-\frac{1}{2M}\right)^{m_2} \left(-\frac{1}{2M}\right)^k \frac{(2k-1)!!}{(-M)^k} \underbrace{\boxed{T \cdots T}}_N \otimes \boxed{0} \otimes \boxed{0} \otimes \boxed{L},
\end{aligned} \tag{8.46}$$

which are exactly consistent with the results obtained by using the decoupling of high energy ZNS in the previous section and the saddle-point calculation which will be discussed in the following section.

C. Saddle-point approximation

In this section, we shall calculate the high energy limits of various scattering amplitudes based on saddle-point approximation. Since the decoupling of ZNS holds true for arbitrary physical processes, in order to check the ratios among scattering amplitudes at the same mass level, it is helpful to choose low-lying states to simplify calculations. For instance, in the case of 4-point amplitudes, we fix the first vertex to be a $M^2 = 0$ photon with polarization vector ϵ^μ (in the -1 ghost picture, and ϕ is the bosonized ghost operator),

$$V_1 \equiv \epsilon^\mu \psi_\mu e^{-\phi} e^{ik_1 X_1}, \quad \epsilon \cdot k_1 = k_1^2 = 0; \tag{8.47}$$

and the third and fourth vertices to be $M^2 = -1$ tachyon (in the 0 ghost picture),

$$V_{3,4} \equiv k_{3,4}^\mu \psi_\mu e^{ik_{3,4} X_{3,4}}, \quad k_{3,4}^2 = -1. \tag{8.48}$$

We shall vary the second vertex at the same level and compare the scattering amplitudes to obtain the proportional constants.

1. $M^2 = 2$

The second vertex operators at mass level $M^2 = 2$, are given by (in the -1 ghost picture),

$$(\alpha_{-1}^T)(b_{-\frac{1}{2}}^T) |0, k\rangle \Rightarrow \psi^T \partial X^T e^{-\phi} e^{ikX}, \tag{8.49}$$

$$(\alpha_{-1}^L)(b_{-\frac{1}{2}}^L) |0, k\rangle \Rightarrow \psi^L \partial X^L e^{-\phi} e^{ikX}, \tag{8.50}$$

$$(b_{-\frac{3}{2}}^L) |0, k\rangle \Rightarrow \partial \psi^L e^{-\phi} e^{ikX}. \tag{8.51}$$

Here we have used the polarization basis to specify the particle spins, e.g., $\psi^T \equiv e_\mu^T \cdot \psi^\mu$.

To illustrate the procedure, we take the first state, Eq.(8.49), as an example to calculate the scattering amplitude among one massive tensor ($M^2 = 2$) with one photon (V_1) and two tachyons (V_3, V_4). As in the case of open bosonic string theory, we list the contributions of $s - t$ channel only. The 4-point function is given by

$$\int_0^1 dx_2 \langle (\psi_1^{T_1} e^{-\phi_1} e^{ik_1 X_1}) (\psi_2^{T_2} \partial X_2^{T_2} e^{-\phi_2} e^{ik_2 X_2}) (k_{3\lambda} \psi_3^\lambda e^{ik_3 X_3}) (k_{4\sigma} \psi_4^\sigma e^{ik_4 X_4}) \rangle, \quad (8.52)$$

where we have suppressed the $SL(2, R)$ gauge-fixed world-sheet coordinates $x_1 = 0, x_3 = 1, x_4 = \infty$. Notice that in both the first and second vertices, it is possible to allow fermion operators ψ^μ to have polarization in transverse direction T_i out of the scattering plane. As we shall see in next section that this leads to a new feature of supersymmetric stringy amplitudes in the high energy limit. At this moment, we only choose the polarization vector to be in the P, L, T directions for a comparison with results obtained by the previous two methods.

A direct application of Wick contraction among fermions ψ , ghosts ϕ , and bosons X leads to the following result

$$\int_0^1 dx \left[\frac{(3, 4)(e^{T_1} \cdot e^{T_2})}{x} - (e^{T_1} \cdot k_3)(e^{T_2} \cdot k_4) + \frac{(e^{T_2} \cdot k_3)(e^{T_1} \cdot k_4)}{1-x} \right] \frac{1}{x} \left[\frac{e^{T_2} \cdot k_3}{1-x} \right] x^{(1,2)} (1-x)^{(2,3)}, \quad (8.53)$$

where we have used the short-hand notation, $(3, 4) \equiv k_3 \cdot k_4$. Based on the kinematic variables and the master formula for saddle-point approximation,

$$\int dx \ u(x) \exp^{-Kf(x)} = u_0 e^{-Kf_0} \sqrt{\frac{2\pi}{Kf_0''}} \left\{ 1 + \left[\frac{u_0''}{2u_0 f_0''} - \frac{u_0' f_0^{(3)}}{2u_0 (f_0'')^2} - \frac{f_0^{(4)}}{8(f_0'')^2} + \frac{5[f_0^{(3)}]^2}{24(f_0'')^3} \right] \frac{1}{K} + O\left(\frac{1}{K^2}\right) \right\}, \quad (8.54)$$

where u_0, f_0, u_0', f_0'' , etc, stand for the values of functions and their derivatives evaluated at the saddle point $f'(x_0) = 0$. In order to apply this master formula to calculate stringy amplitudes, we need the following substitutions ($\alpha' = 1/2$)

$$K \equiv 2E^2, \quad (8.55)$$

$$f(x) \equiv \ln(x) - \tau \ln(1-x), \quad (8.56)$$

$$\tau \equiv -\frac{(2, 3)}{(1, 2)} \rightarrow \sin^2 \frac{\theta}{2}, \quad (8.57)$$

where θ is the scattering angle in center of momentum frame and the saddle point for the integration of moduli is $x_0 = \frac{1}{1-\tau}$. In the first scattering amplitude corresponding to Eq.(8.49), we can identify the $u(x)$ function as

$$u_I(x) \equiv \left[\frac{(3,4)(e^{T_1} \cdot e^{T_2})}{x} - (e^{T_1} \cdot k_3)(e^{T_2} \cdot k_4) + \frac{(e^{T_2} \cdot k_3)(e^{T_1} \cdot k_4)}{1-x} \right] \frac{1}{x} \left[\frac{e^{T_2} \cdot k_3}{1-x} \right]. \quad (8.58)$$

Equipped with this, we obtain the high energy limit of the first amplitude,

$$\begin{aligned} & 2E^2(1-\tau)(e^T \cdot k_3)x_0^{(1,2)-1}(1-x_0)^{(2,3)-1} \sqrt{\frac{\pi\tau}{E^2(1-\tau)^3}} \\ & = 4\sqrt{\pi}E^2(1-\tau)^2x_0^{(1,2)}(1-x_0)^{(2,3)}. \end{aligned} \quad (8.59)$$

Next, we replace the second vertex operator in Eq.(8.52) by Eq.(8.50), and the 4-point function is given by

$$\int_0^1 dx \frac{1}{M^2} \left[(e^T \cdot k_3)(2,4) - \frac{(e^T \cdot k_4)(2,3)}{1-x} \right] \frac{1}{x} \left[\frac{(1,2)}{x} - \frac{(2,3)}{1-x} \right] x^{(1,2)}(1-x)^{(2,3)}. \quad (8.60)$$

Here we can identify the $u(x)$ function for saddle-point master formula, Eq.(8.54)

$$u_{II}(x) \equiv \frac{(e^T \cdot k_3)(1,2)}{M^2x} \left[(2,4) + \frac{(2,3)}{1-x} \right] f'(x). \quad (8.61)$$

One can check that $u_{II}(x_0) = u'_{II}(x_0) = 0$, and

$$u''_{II}(x_0) = \frac{2(1,2)(2,3)(e^T \cdot k_3)}{M^2x(1-x)^2} f''(x_0). \quad (8.62)$$

Thus, the amplitude associated with the massive state, Eq.(8.50), is given by

$$\begin{aligned} & -\frac{2}{M^2}E^2\tau(e^T \cdot k_3)x_0^{(1,2)-1}(1-x_0)^{(2,3)-2} \sqrt{\frac{\pi\tau}{E^2(1-\tau)^3}} \\ & = \frac{4}{M^2}\sqrt{\pi}E^2(1-\tau)^2x_0^{(1,2)}(1-x_0)^{(2,3)}. \end{aligned} \quad (8.63)$$

In the third case, after replacing the second vertex operator in Eq.(8.52) by Eq.(8.51), we get the Wick contraction

$$\int_0^1 dx \frac{1}{M} \left[-\frac{(e^T \cdot k_4)(2,3)}{(1-x)^2} \right] \frac{1}{x} x^{(1,2)}(1-x)^{(2,3)}. \quad (8.64)$$

The high energy limit of this amplitude, after applying the master formula of saddle-point approximation, is

$$\begin{aligned} & \frac{2}{M}E^2\tau(e^T \cdot k_3)x^{(1,2)-1}(1-x)^{(2,3)-2} \sqrt{\frac{\pi\tau}{E^2(1-\tau)^3}} \\ & = -\frac{4}{M}\sqrt{\pi}E^2(1-\tau)^2x_0^{(1,2)}(1-x_0)^{(2,3)}. \end{aligned} \quad (8.65)$$

In conclusion, from these results, Eqs.(8.59),(8.63),(8.65), we find the ratios between the 4-point amplitudes associated with $(\alpha_{-1}^T)(b_{-\frac{1}{2}}^T)|0, k\rangle$, $(\alpha_{-1}^L)(b_{-\frac{1}{2}}^L)|0, k\rangle$, and $(b_{-\frac{3}{2}}^L)|0, k\rangle$ to be $1 : \frac{1}{M^2} : -\frac{1}{M}$, in perfect agreement with Eqs.(8.12),(8.16) and Eq. (8.38).

2. $M^2 = 4$

Our previous examples only involve one fermion operator $b_{-\frac{1}{2}}^T, b_{-\frac{1}{2}}^L, b_{-\frac{3}{2}}^L$. Since in the 4-point functions with the fixed states $V_1 \rightarrow$ photon, $V_{3,4} \rightarrow$ tachyons, the maximum fermion number of the second vertex is three, it is of interest to see the pattern of stringy amplitudes associated with the next massive vertices at $M^2 = 4$.

At this mass level, the relevant states and the vertex operators are (in the -1 ghost picture)

$$(b_{-\frac{1}{2}}^T)(b_{-\frac{1}{2}}^L)(b_{-\frac{3}{2}}^L)|0, k\rangle \Rightarrow \psi^T \psi^L \partial \psi^L e^{-\phi} e^{ikX}, \quad (8.66)$$

$$(\alpha_{-1}^T)(\alpha_{-1}^T)(b_{-\frac{1}{2}}^T)|0, k\rangle \Rightarrow \psi^T \partial X^T \partial X^T e^{-\phi} e^{ikX}. \quad (8.67)$$

To calculate 4-point functions, we can fix the first vertex (V_1) to be a photon state in the -1 ghost picture, Eq.(8.47), and the third and the fourth vertices to be tachyon state in the 0 ghost picture, Eq.(8.48).

Since the applications of saddle-point approximation is essentially identical to previous cases, we simply list the results of our calculations

$$\begin{aligned} & (b_{-\frac{1}{2}}^T)(b_{-\frac{1}{2}}^L)(b_{-\frac{3}{2}}^L)|0, k\rangle \\ & \Rightarrow \int_0^1 dx_2 \langle (\psi_1^{T_1} e^{-\phi_1} e^{ik_1 X_1}) (\psi_2^{T_2} \psi_2^{L_2} \partial \psi_2^{L_2} e^{-\phi_2} e^{ik_2 X_2}) (k_{3\lambda} \psi_3^\lambda e^{ik_3 X_3}) (k_{4\sigma} \psi_4^\sigma e^{ik_4 X_4}) \rangle \\ & = \frac{4\sqrt{\pi}}{M^2} E^3 \tau^{-\frac{1}{2}} (1-\tau)^{\frac{7}{2}} x_0^{(1,2)} (1-x_0)^{(2,3)}, \end{aligned} \quad (8.68)$$

$$\begin{aligned} & (\alpha_{-1}^T)(\alpha_{-1}^T)(b_{-\frac{1}{2}}^T)|0, k\rangle \\ & \Rightarrow \int_0^1 dx_2 \langle (\psi_1^{T_1} e^{-\phi_1} e^{ik_1 X_1}) (\psi_2^{T_2} X_2^{L_2} X_2^{L_2} e^{-\phi_2} e^{ik_2 X_2}) (k_{3\lambda} \psi_3^\lambda e^{ik_3 X_3}) (k_{4\sigma} \psi_4^\sigma e^{ik_4 X_4}) \rangle \\ & = 8\sqrt{\pi} E^3 \tau^{-\frac{1}{2}} (1-\tau)^{\frac{7}{2}} x_0^{(1,2)} (1-x_0)^{(2,3)}. \end{aligned} \quad (8.69)$$

Combining these results, we conclude that the ratio between the $M^2 = 4$ vertices is given by

$$(b_{-\frac{1}{2}}^T)(b_{-\frac{1}{2}}^L)(b_{-\frac{3}{2}}^L)|0, k\rangle : (\alpha_{-1}^T)(\alpha_{-1}^T)(b_{-\frac{1}{2}}^L)|0, k\rangle = \frac{1}{M^2} : 2 = 1 : 8, \quad (8.70)$$

which agrees with Eq.(8.20).

3. GSO odd vertices at $M^2 = 5$

In addition to the stringy amplitudes associated with GSO even vertices we have calculated in the previous sections, we can also apply the same method to those associated with the GSO odd vertices. While it is a common practice to project out the GSO odd states in order to maintain spacetime supersymmetry, it turns out that we do find linear relation among these amplitudes. This seems to suggest a hidden structure of superstring theory in the high energy limit.

To see this, we examine the vertices of odd GSO parity, at the mass level $M^2 = 5$. Based on the power-counting rule as in the bosonic string case, we can identify the relevant vertices and the associated vertex operators as follows

$$(\alpha_{-1}^T)(b_{-\frac{1}{2}}^T)(b_{-\frac{3}{2}}^L) |0, k\rangle \Rightarrow \psi^T \partial \psi^L \partial X^T e^{-\phi} e^{ikX}, \quad (8.71)$$

$$(\alpha_{-1}^L)(b_{-\frac{1}{2}}^L)(b_{-\frac{3}{2}}^L) |0, k\rangle \Rightarrow \psi^L \partial \psi^L \partial X^L e^{-\phi} e^{ikX}. \quad (8.72)$$

To calculate 4-point functions, we can fix the first vertex (V_1) to be a tachyon state in the -1 ghost picture,

$$V_1 = e^{-\phi_1} e^{ik_1 \cdot X_1}, \quad (8.73)$$

and the third and the fourth vertices to be tachyon state in the 0 ghost picture, as Eq.(8.48).

Since the applications of saddle-point approximation is essentially identical to previous cases, we simply list the results of our calculations

$$\begin{aligned} & (\alpha_{-1}^T)(b_{-\frac{1}{2}}^T)(b_{-\frac{3}{2}}^L) |0, k\rangle \\ & \Rightarrow \int_0^1 dx_2 \langle (e^{-\phi_1} e^{ik_1 X_1}) (\psi_2^{T_2} \partial \psi_2^{L_2} \partial X_2^{T_2} e^{-\phi_2} e^{ik_2 X_2}) (k_{3\lambda} \psi_3^\lambda e^{ik_3 X_3}) (k_{4\sigma} \psi_4^\sigma e^{ik_4 X_4}) \rangle, \\ & = -\frac{8\sqrt{\pi}}{M} E^3 \tau^{-\frac{1}{2}} (1-\tau)^{\frac{7}{2}} x_0^{(1,2)} (1-x_0)^{(2,3)}, \end{aligned} \quad (8.74)$$

$$\begin{aligned} & (\alpha_{-1}^L)(b_{-\frac{1}{2}}^L)(b_{-\frac{3}{2}}^L) |0, k\rangle \\ & \Rightarrow \int_0^1 dx_2 \langle (e^{-\phi_1} e^{ik_1 X_1}) (\psi_2^{L_2} \partial \psi_2^{L_2} \partial X_2^{L_2} e^{-\phi_2} e^{ik_2 X_2}) (k_{3\lambda} \psi_3^\lambda e^{ik_3 X_3}) (k_{4\sigma} \psi_4^\sigma e^{ik_4 X_4}) \rangle, \\ & = -\frac{4\sqrt{\pi}}{M^3} E^3 \tau^{-\frac{1}{2}} (1-\tau)^{\frac{7}{2}} x_0^{(1,2)} (1-x_0)^{(2,3)}. \end{aligned} \quad (8.75)$$

It is worth noting that in the second calculations, we need to include both $u''(x_0)$ and $u^{(3)}(x_0)$ terms of the first order corrections in saddle-point approximation, Eq.(8.54), to get the correct answer.

Combining these results, we conclude that the ratio between the $M^2 = 3$ vertices is given by

$$(\alpha_{-1}^T)(b_{-\frac{1}{2}}^T)(b_{-\frac{3}{2}}^L) |0, k\rangle : (\alpha_{-1}^L)(b_{-\frac{1}{2}}^L)(b_{-\frac{3}{2}}^L) |0, k\rangle = 2M^2 : 1 = 10 : 1. \quad (8.76)$$

Notice that here we also find an interesting connection between GSO even $M^2 = 4$ amplitudes and those of GSO odd parity at $M^2 = 5$. The high energy limits of the four amplitudes, Eqs.(8.68),(8.69),(8.74),(8.75), are proportional to each other. and their ratios are $\sqrt{5} : 8\sqrt{5} : (-8) : -\frac{4}{5}$.

D. Polarizations orthogonal to the scattering plane

In this section we consider high energy scattering amplitudes of string states with polarizations $e_{T^i}, i = 3, 4, \dots, 25$, orthogonal to the scattering plane. We will present some examples with saddle-point calculations and compare them with those calculated in the previous section. We will find that they are all proportional to the previous ones considered before. These scattering amplitudes are of subleading order in energy for the case of $26D$ open bosonic string theory. The existence of these new high energy scattering amplitudes is due to the worldsheet fermion exchange in the correlation functions as we will see in the following examples. Our first example is to consider Eq.(8.52) and replace $\psi_1^{T_1}$ and $\psi_2^{T_2}$ by $\psi_1^{T_i}$ and $\psi_2^{T_i^2}$ respectively

$$\int_0^1 dx_2 \langle (\psi_1^{T_i} e^{-\phi_1} e^{ik_1 X_1}) (\psi_2^{T_i^2} \partial X_2^{T_2} e^{-\phi_2} e^{ik_2 X_2}) (k_{3\lambda} \psi_3^\lambda e^{ik_3 X_3}) (k_{4\sigma} \psi_4^\sigma e^{ik_4 X_4}) \rangle. \quad (8.77)$$

The calculation of Eq.(8.77) is similar to that of Eq.(8.52) except that, for this new case, one ends up with only the first term in Eq.(8.53), and the second and the third terms vanish. Remarkably, the final answer is

$$\begin{aligned} & -2E^2(1-\tau)(e^T \cdot k_3)x_0^{(1,2)-1}(1-x_0)^{(2,3)-1}\sqrt{\frac{\pi\tau}{E^2(1-\tau)^3}} \\ & = -4\sqrt{\pi}E^2(1-\tau)^2x_0^{(1,2)}(1-x_0)^{(2,3)}, \end{aligned} \quad (8.78)$$

which is proportional to Eq.(8.59). Our second example is again to replace $\psi_1^{T_1}$ and $\psi_2^{T_2}$ in Eq.(8.68) by $\psi_1^{T_1^i}$ and $\psi_2^{T_2^i}$ respectively. One gets exactly the same answer as Eq.(8.68). The two examples above seem to suggest that high energy scattering of string states with polarizations e_{T^i} are the same as that of polarization e_T up to a sign. Let's consider the third example to justify this point. It is straightforward to show the following

$$\begin{aligned} & \int_0^1 dx_2 \langle (\psi_1^L \psi_1^{T_1} \psi_1^{T_1^i} e^{-\phi_1} e^{ik_1 X_1}) (\psi_2^L \psi_2^{T_2} \psi_2^{T_2^i} \partial X_2^{T_2} e^{-\phi_2} e^{ik_2 X_2}) (k_{3\lambda} \psi_3^\lambda e^{ik_3 X_3}) (k_{4\sigma} \psi_4^\sigma e^{ik_4 X_4}) \rangle \\ &= N[4E^4(1-\tau) - 4E^4(1-\tau)^2 - 4E^4\tau(1-\tau)] = 0 \end{aligned} \quad (8.79)$$

On the other hand, if we assume the symmetry for all transverse polarization vectors T, T^i in the scattering amplitudes, one can easily derive the same conclusion without detailed calculations. Since replacing T^i polarization vectors of both vertices in Eq.(8.79) by T will naturally leads to a null result due to anti-commuting property of fermions.

It is clear from the above calculations that the existence of these new high energy scattering amplitudes of string states with polarizations e_{T^i} orthogonal to the scattering plane is due to the worldsheet fermion exchange in the correlation functions. These fermion exchanges do not exist in the bosonic string correlation functions and is, presumably, related to the high energy massive spacetime fermionic scattering amplitudes in the R-sector of the theory.

Physically, the high energy scattering amplitudes of spacetime fermion contain the symmetry of rotations among different polarizations in the spin space and our results here seem to justify this observation. If this conjecture turns out to be true, then the list of vertices we considered in Eq.(8.7) to Eq.(8.10) for high energy stringy amplitudes should be extended and includes the cases with $b_{-\frac{1}{2}}^T$ replaced by $b_{-\frac{1}{2}}^{T^i}$. Obviously, these new high energy amplitudes create complications for a full understanding of stringy symmetry. Nevertheless, the claim that there is only one independent high energy scattering amplitude at each fixed mass level of the string spectrum persists in the case of superstring theory, at least, for the NS sector of the theory.

In conclusion of this chapter we have explicitly calculated all high energy scattering amplitudes of string states with polarizations on the scattering plane of open superstring theory. In particular, the proportionality constants among high energy scattering amplitudes of different string states at each fixed but arbitrary mass level are determined by using three

different methods. These constants are shown to originate from ZNS in the spectrum as in the case of open bosonic string theory.

In addition, we discover new high energy scattering amplitudes, which are still proportional to the previous ones, with polarizations *orthogonal* to the scattering plane. We conjecture the existence of a symmetry among high energy scattering amplitudes with polarizations e_{Ti} and e_T . These scattering amplitudes are subleading order in energy for the case of open bosonic string theory. The existence of these new high energy scattering amplitudes is due to the worldsheet fermion exchange in the correlation functions and is argued to be related to the high energy massive spacetime fermionic scattering amplitudes in the R-sector of the theory. Finally, our study also suggests that the nature of GSO projection in superstring theory might be simplified in the high energy limit. Hopefully, this is in connection with the conjecture that supersymmetry is realized in broken phase without GSO projection in the open string theory [133, 134].

It would be of crucial importance to calculate high energy massive fermion scattering amplitudes in the R-sector to complete the proof of Gross's two conjectures on high energy symmetry of superstring theory. The construction of general *massive* spacetime fermion vertex, involving picture changing, will be the first step toward understanding of the high energy behavior of superstring theory.

IX. HARD STRING SCATTERINGS FROM D-BRANES/O-PLANES

In this chapter, we study scatterings of bosonic closed strings from D-branes [41] in section A, and O-planes [46] in section B. In particular, we will discuss hard strings scattered from D-particle [41] and D-domain-wall [45]. We will also study hard strings scattered from O-particle [46] and O-domain-wall [46]. In addition, in section C, we calculate the absorption amplitudes [126] of a closed string state at arbitrary mass level leading to two open string states on the D-brane at high energies.

A. Scatterings from D-branes

In this section we study the general structure of an arbitrary incoming closed string state scatters from D-brane and ends up with an arbitrary spin outgoing closed string states

at arbitrary mass levels [41]. The scattering of massless string states from D-brane has been well studied in the literature and can be found in [52, 135–139]. Here we extend the calculations of massless closed string states to massive closed string states at arbitrary mass levels. Since the mass of D-brane scales as the inverse of the string coupling constant $1/g$, we will assume that it is infinitely heavy to leading order in g and does not recoil. We will first show that, for the $(0 \rightarrow 1)$ and $(1 \rightarrow \infty)$ channels, all the scattering amplitudes can be expressed in terms of the beta functions, thanks to the *momentum conservation on the D-brane*.

Alternatively, the Kummer relation of the hypergeometric function ${}_2F_1$ can be used to reduce the scattering amplitudes to the usual beta function [41]. After summing up the $(0 \rightarrow 1)$ and $(1 \rightarrow \infty)$ channels, we discover that all the scattering amplitudes can be expressed in terms of the generalized hypergeometric function ${}_3F_2$ with special arguments, which terminates to a finite sum and, as a result, the whole scattering amplitudes consistently reduce to the usual beta function.

For the simple case of D-particle, we explicitly calculate [41] high energy limit of a set of scattering amplitudes for arbitrary mass levels, and derive infinite linear relations among them for each fixed mass level. Since the calculation of decoupling of high energy ZNS remains the same as the case of scatterings without D-brane, the ratios of these high energy scattering amplitudes are found to be consistent with the decoupling of high energy ZNS in Chapter V. The cases of RR strings scattered from D-particle will be discussed in chapter XIV where the complete ratios among GR scattering amplitudes will be calculated.

We will first begin with the simple case of tachyon to tachyon scattering and then generalize to scatterings of states at arbitrary mass levels. The standard propagators of the left and right moving fields are

$$\langle X^\mu(z) X^\nu(w) \rangle = -\eta^{\mu\nu} \log(z - w), \quad (9.1)$$

$$\langle \tilde{X}^\mu(\bar{z}) \tilde{X}^\nu(\bar{w}) \rangle = -\eta^{\mu\nu} \log(\bar{z} - \bar{w}). \quad (9.2)$$

In addition, there are also nontrivial correlator between the right and left moving fields as well

$$\langle X^\mu(z) \tilde{X}^\nu(\bar{w}) \rangle = -D^{\mu\nu} \log(z - \bar{w}) \quad (9.3)$$

as a result of the boundary condition at the real axis. Propagator Eq.(9.3) has the standard form Eq.(9.1) for the fields satisfying Neumann boundary condition, while matrix D reverses

the sign for the fields satisfying Dirichlet boundary condition. We will follow the standard notation and make the following replacement

$$\tilde{X}^\mu(\bar{z}) \rightarrow D^\mu{}_\nu X^\nu(\bar{z}) \quad (9.4)$$

which allows us to use the standard correlators Eq.(9.1) throughout our calculations. As we will see, the existence of the Propagator Eq.(9.3) has far-reaching effect on the string scatterings from D-brane.

1. Tachyon to tachyon

In this section, we consider the tachyon to tachyon scattering amplitude

$$\begin{aligned} A_{tach} &= \int d^2 z_1 d^2 z_2 \langle V_1(z_1, \bar{z}_1) V_2(z_2, \bar{z}_2) \rangle \\ &= \int d^2 z_1 d^2 z_2 \langle V(k_1, z_1) \tilde{V}(k_1, \bar{z}_1) V(k_2, z_2) \tilde{V}(k_2, \bar{z}_2) \rangle \\ &= \int d^2 z_1 d^2 z_2 \langle e^{ik_1 X(z_1)} e^{ik_1 \tilde{X}(\bar{z}_1)} e^{ik_2 X(z_2)} e^{ik_2 \tilde{X}(\bar{z}_2)} \rangle \\ &= \int d^2 z_1 d^2 z_2 (z_1 - \bar{z}_1)^{k_1 \cdot D \cdot k_1} (z_2 - \bar{z}_2)^{k_2 \cdot D \cdot k_2} |z_1 - z_2|^{2k_1 \cdot k_2} |z_1 - \bar{z}_2|^{2k_1 \cdot D \cdot k_2}. \end{aligned} \quad (9.5)$$

To fix the $SL(2, R)$ invariance, we set $z_1 = iy$ and $z_2 = i$. Introducing the $SL(2, R)$ Jacobian

$$d^2 z_1 d^2 z_2 = 4(1 - y^2) dy, \quad (9.6)$$

we have, for the $(0 \rightarrow 1)$ channel,

$$\begin{aligned} A_{tach}^{(0 \rightarrow 1)} &= 4(2i)^{k_1 \cdot D \cdot k_1 + k_2 \cdot D \cdot k_2} \int_0^1 dy y^{k_2 \cdot D \cdot k_2} (1 - y)^{2k_1 \cdot k_2 + 1} (1 + y)^{2k_1 \cdot D \cdot k_2 + 1} \\ &= 4(2i)^{2a_0} \int_0^1 dy y^{a_0} (1 - y)^{b_0} (1 + y)^{c_0} \\ &= 4(2i)^{2a_0} \frac{\Gamma(a_0 + 1) \Gamma(b_0 + 1)}{\Gamma(a_0 + b_0 + 2)} {}_2F_1(-c_0, a_0 + 1, a_0 + b_0 + 2, -1) \end{aligned} \quad (9.7)$$

$$\begin{aligned} &= 4(2i)^{2a_0} \frac{\Gamma(a_0 + 1) \Gamma(b_0 + 1)}{\Gamma(a_0 + b_0 + 2)} 2^{-2a_0 - 1 - N} {}_2F_1(N - a_0, b_0 + 1, a_0 + b_0 + 2, -1) \\ &= 4(2i)^{2a_0} 2^{-2a_0 - 1 - N} \int_0^1 dt t^{b_0} (1 - t)^{a_0} (1 + t)^{a_0 + N}. \end{aligned} \quad (9.8)$$

In the above calculations, we have defined

$$a_0 = k_1 \cdot D \cdot k_1 = k_2 \cdot D \cdot k_2, \quad (9.9)$$

$$b_0 = 2k_1 \cdot k_2 + 1, \quad (9.10)$$

$$c_0 = 2k_1 \cdot D \cdot k_2 + 1, \quad (9.11)$$

so that

$$2a_0 + b_0 + c_0 + 2 = 4N_1 \equiv -N, \quad (9.12)$$

and $-k_1^2 = M^2 \equiv \frac{M_{closed}^2}{2\alpha'_{closed}} = 2(N_1 - 1)$, $N_1 = 0$ for tachyon. We have also used the integral representation of the hypergeometric function

$${}_2F_1(\alpha, \beta, \gamma, z) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \int_0^1 dt t^{\beta-1} (1-t)^{\gamma-\beta-1} (1-zt)^{-\alpha}, \quad (9.13)$$

and the following identity

$${}_2F_1(\alpha, \beta, \gamma; x) = (1-x)^{\gamma-\alpha-\beta} {}_2F_1(\gamma-\alpha, \gamma-\beta, \gamma; x), \quad (9.14)$$

which we discuss in the section IX A 5. In addition, the momentum conservation on the D-brane

$$D \cdot k_1 + k_1 + D \cdot k_2 + k_2 = 0 \quad (9.15)$$

is crucial to get the final result Eq.(9.8). Finally, by using change of variable $\tilde{t} = t^2$, Eq.(9.8) can be further reduced to the beta function

$$A_{tach}^{(0 \rightarrow 1)} \simeq \frac{\Gamma(a_0 + 1) \Gamma\left(\frac{b_0 + 1}{2}\right)}{\Gamma\left(a_0 + \frac{b_0}{2} + \frac{3}{2}\right)} = B\left(a_0 + 1, \frac{b_0 + 1}{2}\right) \quad (9.16)$$

where we have omitted an irrelevant factor.

For the $(1 \rightarrow \infty)$ channel, we use the change of variable $y = \frac{1+t}{1-t}$ and end up with the same result

$$\begin{aligned} A_{tach}^{(1 \rightarrow \infty)} &= 4(2i)^{k_1 \cdot D \cdot k_1 + k_2 \cdot D \cdot k_2} \int_1^\infty dy y^{k_2 \cdot D \cdot k_2} (y-1)^{2k_1 \cdot k_2 + 1} (1+y)^{2k_1 \cdot D \cdot k_2 + 1} \\ &= 4(2i)^{2a_0} 2^{-2a_0 - 1 - N} \int_0^1 dt t^{b_0} (1-t)^{a_0 + N} (1+t)^{a_0} \\ &\simeq \frac{\Gamma(a_0 + 1) \Gamma\left(\frac{b_0 + 1}{2}\right)}{\Gamma\left(a_0 + \frac{b_0}{2} + \frac{3}{2}\right)} = B\left(a_0 + 1, \frac{b_0 + 1}{2}\right) \end{aligned} \quad (9.17)$$

since $N = 0$ for the case of tachyon.

Alternatively, one can use the Kummer formula of hypergeometric function

$${}_2F_1(\alpha, \beta, 1 + \alpha - \beta, -1) = \frac{\Gamma(1 + \alpha - \beta)\Gamma(1 + \frac{\alpha}{2})}{\Gamma(1 + \alpha)\Gamma(1 + \frac{\alpha}{2} - \beta)} \quad (9.18)$$

and

$$\Gamma\left(\frac{1 + \alpha}{2}\right) = \frac{2^{-\alpha}\sqrt{\pi}\Gamma(1 + \alpha)}{\Gamma(1 + \frac{\alpha}{2})}, \quad (9.19)$$

to reduce Eq.(9.7) to the final result Eq.(9.16). In this calculation, we have used the *Kummer condition*

$$\gamma = 1 + \alpha - \beta, \quad (9.20)$$

which is equivalent to the momentum conservation on the D-brane Eq.(9.15).

2. Tensor to tensor

In this section, we generalize the previous calculation to general tensor to tensor scatterings. In this case, we define

$$a = k_1 \cdot D \cdot k_1 + n_a \equiv a_0 + n_a, \quad (9.21)$$

$$b = 2k_1 \cdot k_2 + 1 + n_b \equiv b_0 + n_b, \quad (9.22)$$

$$c = 2k_1 \cdot D \cdot k_2 + 1 + n_c \equiv c_0 + n_c, \quad (9.23)$$

where n_a , n_b and n_c are integer and

$$N' = -(2n_a + n_b + n_c),$$

so that

$$2a + b + c + 2 + N' = 4N_1 \implies 2a + b + c + 2 = 4N_1 - N' \equiv -N \quad (9.24)$$

where $k_1^2 = 2(N_1 - 1)$ and N_1 is now the mass level of k_1 . After a similar calculation as the previous section, it is easy to see that a typical term in the expression of the general tensor to tensor scattering amplitudes can be reduced to the following integral

$$\begin{aligned}
I_{(0 \rightarrow 1)} &= \int_0^1 dt t^a (1-t)^b (1+t)^c, \\
&= \frac{\Gamma(a+1) \Gamma(b+1)}{\Gamma(a+b+2)} {}_2F_1(-c, a+1, a+b+2, -1) \\
&= 2^{b+c+1} \frac{\Gamma(a+1) \Gamma(b+1)}{\Gamma(a+b+2)} {}_2F_1(-a-N, b+1, a+b+2, -1) \\
&= 2^{-2a-1-N} \int_0^1 dt t^b (1-t)^a (1+t)^{a+N}.
\end{aligned} \tag{9.25}$$

Similarly, for the $(1 \rightarrow \infty)$ channel, one gets

$$\begin{aligned}
I_{(1 \rightarrow \infty)} &= \int_1^\infty dy y^a (y-1)^b (1+y)^c \\
&= 2^{-2a-1-N} \int_0^1 dt t^b (1-t)^{a+N} (1+t)^a.
\end{aligned} \tag{9.26}$$

The sum of the two channels gives

$$\begin{aligned}
I &= I_{(0 \rightarrow 1)} + I_{(1 \rightarrow \infty)} \\
&= 2^{-2a-1-N} \int_0^1 dt t^b (1-t)^a (1+t)^a \left[(1+t)^N + (1-t)^N \right] \\
&= 2^{-2a-1-N} \sum_{m=0}^N [1 + (-1)^m] \binom{N}{m} \int_0^1 dt t^{b+m} (1-t)^a (1+t)^a \\
&= 2^{-2a-2-N} \sum_{m=0}^N [1 + (-1)^m] \binom{N}{m} \cdot \frac{\Gamma(a+1) \Gamma(\frac{b+1}{2} + \frac{m}{2})}{\Gamma(a + \frac{b+3}{2} + \frac{m}{2})} \\
&= 2^{-2a-1-N} \frac{\Gamma(a+1) \Gamma(\frac{b+1}{2})}{\Gamma(a + \frac{b+3}{2})} \sum_{n=0}^{\lfloor \frac{N}{2} \rfloor} \binom{N}{2n} \frac{(\frac{b+1}{2})_n}{(a + \frac{b+3}{2})_n} \\
&= 2^{-2a-1-N} \cdot B\left(a+1, \frac{b+1}{2}\right) \cdot {}_3F_2\left(\frac{b+1}{2}, -\left[\frac{N}{2}\right], \frac{1}{2} - \left[\frac{N}{2}\right]; a + \frac{b+3}{2}, \frac{1}{2}; 1\right)
\end{aligned} \tag{9.27}$$

where the generalized hypergeometric function ${}_3F_2$ is defined to be

$${}_3F_2(\alpha_1, \alpha_2, \alpha_3; \gamma_1, \gamma_2; x) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n (\alpha_2)_n (\alpha_3)_n}{(\gamma_1)_n (\gamma_2)_n} \frac{x^n}{n!}. \tag{9.28}$$

Note that the energy dependence of the prefactor $4(2i)^{k_1 \cdot D \cdot k_1 + k_2 \cdot D \cdot k_2}$ in the scattering amplitude cancels, apart from an irrelevant factor, the energy dependence of $2^{-2a-1-N}$ by using Eq.(9.24). For $N = 0$, one recovers the result of tachyon scattering amplitude Eq.(9.8). For the special arguments of ${}_3F_2$ in Eq.(9.27), the hypergeometric function *terminates to a finite sum* and, as a result, the whole scattering amplitudes consistently reduce to the usual beta function. The explicit forms of ${}_3F_2$ for some integer N are given in the section IX A 5.

3. Hard strings scattered from D-particle

In this section, we will calculate the high energy limit of string scattered from D-brane [41]. In particular, we will calculate the ratios among scattering amplitudes of different string states at high energies. For our purpose here, for simplicity, we will only consider the string states with the form of Eq.(5.67) with $m = 0$ (the ratios for the $m \neq 0$ case will be discussed in chapter XIV). The reason is as following. It was shown that [26–29] the leading order amplitudes containing this component will drop from energy order E^{4m} to E^{2m} , and one needs to calculate the complicated naive subleading contraction terms between ∂X and ∂X for the *multi*-tensor scattering in order to get the real leading order scattering amplitudes. For our *closed* string scattering calculation here, even for the case of one tachyon and one tensor scattering, one encounters the similar complicated nonzero contraction terms in Eq.(9.3) due to the D-brane. So we will omit high energy scattering amplitudes of string states containing this $(\alpha_{-1}^L)^{2m}$ component. On the other hand, we will also need the result that the high energy closed string ratios are the tensor product of two pieces of open string ratios [35].

To simplify the kinematics, we consider the case of D0 brane or D-particle scatterings [41]. The momentum of the incident particle k_2 is along the $-X$ direction and particle k_1 is scattered at an angle ϕ . We will consider the general case of an incoming tensor state $(\alpha_{-1}^T)^{n-2q} (\alpha_{-2}^L)^q \otimes (\tilde{\alpha}_{-1}^T)^{n-2q'} (\tilde{\alpha}_{-2}^L)^{q'} |0\rangle$ and an outgoing tachyon state. Our result can be easily generalized to the more general two tensor cases. The kinematic setup is

$$e^P = \frac{1}{M} (-E, -k_2, 0) = \frac{k_2}{M}, \quad (9.29)$$

$$e^L = \frac{1}{M} (-k_2, -E, 0), \quad (9.30)$$

$$e^T = (0, 0, 1), \quad (9.31)$$

$$k_1 = (E, k_1 \cos \phi, -k_1 \sin \phi), \quad (9.32)$$

$$k_2 = (-E, -k_2, 0). \quad (9.33)$$

For the scattering of D-particle $D_{ij} = -\delta_{ij}$, and it is easy to calculate

$$e^T \cdot k_2 = e^L \cdot k_2 = 0, \quad (9.34)$$

$$e^T \cdot k_1 = -k_1 \sin \phi \sim -E \sin \phi, \quad (9.35)$$

$$e^T \cdot D \cdot k_1 = k_1 \sin \phi \sim E \sin \phi, \quad (9.36)$$

$$e^T \cdot D \cdot k_2 = 0, \quad (9.37)$$

$$e^L \cdot k_1 = \frac{1}{M} [k_2 E - k_1 E \cos \phi] \sim \frac{E^2}{M} (1 - \cos \phi), \quad (9.38)$$

$$e^L \cdot D \cdot k_1 = \frac{1}{M} [k_2 E + k_1 E \cos \phi] \sim \frac{E^2}{M} (1 + \cos \phi), \quad (9.39)$$

$$e^L \cdot D \cdot k_2 = \frac{1}{M} [-k_2 E - k_1 E] \sim -\frac{2E^2}{M}, \quad (9.40)$$

and

$$a_0 = k_1 \cdot D \cdot k_1 = -E^2 - k_1^2 \sim -2E^2, \quad (9.41)$$

$$b_0 = 2k_1 \cdot k_2 + 1 = 2(E^2 - k_1 k_2 \cos \phi) + 1 \sim 2E^2 (1 - \cos \phi), \quad (9.42)$$

$$c_0 = 2k_1 \cdot D \cdot k_2 + 1 = 2(E^2 + k_1 k_2 \cos \phi) + 1 \sim 2E^2 (1 + \cos \phi). \quad (9.43)$$

The high energy scattering amplitude is then calculated to be

$$\begin{aligned} A_{D-Par} &= \varepsilon_{T^{n-2q}L^q, T^{n-2q'}L^{q'}} \int d^2 z_1 d^2 z_2 \left\langle V_1(z_1, \bar{z}_1) V_2^{T^{n-2q}L^q, T^{n-2q'}L^{q'}}(z_2, \bar{z}_2) \right\rangle \\ &= \varepsilon_{T^{n-2q}L^q, T^{n-2q'}L^{q'}} \int d^2 z_1 d^2 z_2 \cdot \left\langle e^{ik_1 X}(z_1) e^{ik_1 \bar{X}}(\bar{z}_1) \right. \\ &\quad \left. (\partial X^T)^{n-2q} (i\partial^2 X^L)^q e^{ik_2 X}(z_2) (\bar{\partial} \tilde{X}^T)^{n-2q'} (\bar{i}\partial^2 \tilde{X}^L)^{q'} e^{ik_2 \bar{X}}(\bar{z}_2) \right\rangle \\ &= (-1)^{q+q'} \int d^2 z_1 d^2 z_2 (z_1 - \bar{z}_1)^{k_1 \cdot D \cdot k_1} (z_2 - \bar{z}_2)^{k_2 \cdot D \cdot k_2} |z_1 - z_2|^{2k_1 \cdot k_2} |z_1 - \bar{z}_2|^{2k_1 \cdot D \cdot k_2} \\ &\quad \cdot \left[\frac{ie^T \cdot k_1}{z_1 - z_2} + \frac{ie^T \cdot D \cdot k_1}{\bar{z}_1 - z_2} + \frac{ie^T \cdot D \cdot k_2}{\bar{z}_2 - z_2} \right]^{n-2q} \\ &\quad \cdot \left[\frac{ie^T \cdot D \cdot k_1}{z_1 - \bar{z}_2} + \frac{ie^T \cdot k_1}{\bar{z}_1 - \bar{z}_2} + \frac{ie^T \cdot D \cdot k_2}{z_2 - \bar{z}_2} \right]^{n-2q'} \\ &\quad \cdot \left[\frac{e^L \cdot k_1}{(z_1 - z_2)^2} + \frac{e^L \cdot D \cdot k_1}{(\bar{z}_1 - z_2)^2} + \frac{e^L \cdot D \cdot k_2}{(\bar{z}_2 - z_2)^2} \right]^q \\ &\quad \cdot \left[\frac{e^L \cdot D \cdot k_1}{(z_1 - \bar{z}_2)^2} + \frac{e^L \cdot k_1}{(\bar{z}_1 - \bar{z}_2)^2} + \frac{e^L \cdot D \cdot k_2}{(z_2 - \bar{z}_2)^2} \right]^{q'}. \end{aligned} \quad (9.44)$$

Set $z_1 = iy$ and $z_2 = i$ to fix the $SL(2, R)$ invariance, we have

$$\begin{aligned}
A_{D-P_{ar}}^{(0 \rightarrow 1)} &= 4 (2i)^{k_1 \cdot D \cdot k_1 + k_2 \cdot D \cdot k_2} \int_0^1 dy y^{k_1 \cdot D \cdot k_1} (1-y)^{2k_1 \cdot k_2 + 1} (1+y)^{2k_1 \cdot D \cdot k_2 + 1} \\
&\cdot \left[-\frac{e^T \cdot k_1}{1-y} - \frac{e^T \cdot D \cdot k_1}{1+y} - \frac{e^T \cdot D \cdot k_2}{2} \right]^{n-2q} \\
&\cdot \left[\frac{e^T \cdot D \cdot k_1}{1+y} + \frac{e^T \cdot k_1}{1-y} + \frac{e^T \cdot D \cdot k_2}{2} \right]^{n-2q'} \\
&\cdot \left[\frac{e^L \cdot k_1}{(1-y)^2} + \frac{e^L \cdot D \cdot k_1}{(1+y)^2} + \frac{e^L \cdot D \cdot k_2}{4} \right]^q \\
&\cdot \left[\frac{e^L \cdot D \cdot k_1}{(1+y)^2} + \frac{e^L \cdot k_1}{(1-y)^2} + \frac{e^L \cdot D \cdot k_2}{4} \right]^{q'} \\
&= (-1)^n 4 (2i)^{k_1 \cdot D \cdot k_1 + k_2 \cdot D \cdot k_2} (E \sin \phi)^{2n-2(q+q')} \left(\frac{E^2}{M} \right)^{q+q'} \\
&\cdot \int_0^1 dy y^{k_1 \cdot D \cdot k_1} (1-y)^{2k_1 \cdot k_2 + 1} (1+y)^{2k_1 \cdot D \cdot k_2 + 1} \\
&\cdot \left[\frac{1}{1-y} - \frac{1}{1+y} \right]^{2n-2(q+q')} \cdot \left[\frac{1 - \cos \phi}{(1-y)^2} + \frac{1 + \cos \phi}{(1+y)^2} - \frac{1}{2} \right]^{q+q'} \\
&= (-1)^n 4 (2i)^{k_1 \cdot D \cdot k_1 + k_2 \cdot D \cdot k_2} (2E \sin \phi)^{2n} \left(-\frac{1}{8M \sin^2 \phi} \right)^{q+q'} \\
&\cdot \sum_{i=0}^{q+q'} \sum_{j=0}^i \binom{q+q'}{i} \binom{i}{j} (-2)^i (1 - \cos \phi)^j (1 + \cos \phi)^{i-j} \\
&\int_0^1 dy y^{k_1 \cdot D \cdot k_1} (1-y)^{2k_1 \cdot k_2 + 1} (1+y)^{2k_1 \cdot D \cdot k_2 + 1} \\
&\cdot \left[\frac{y}{(1-y)(1+y)} \right]^{2n-2(q+q')} \left[\frac{1}{1-y} \right]^{2j} \left[\frac{1}{1+y} \right]^{2(i-j)}. \tag{9.45}
\end{aligned}$$

Now in the high energy limit, the master formula Eq.(9.27) reduces to

$$\begin{aligned}
I &= I_{(0 \rightarrow 1)} + I_{(1 \rightarrow \infty)} \\
&\simeq 2^{-2a-2-N} B \left(a+1, \frac{b+1}{2} \right) \left[\left(1 + \sqrt{\left| \frac{b}{2a+b} \right|} \right)^N + \left(1 - \sqrt{\left| \frac{b}{2a+b} \right|} \right)^N \right] \\
&\equiv 2^{-2a-2-N} B \left(a+1, \frac{b+1}{2} \right) F_N, \tag{9.46}
\end{aligned}$$

where

$$n_a = 2n - 2(q + q'), \quad (9.47)$$

$$n_b = -2n + 2(q + q') - 2j, \quad (9.48)$$

$$n_c = -2n + 2(q + q') - 2(i - j), \quad (9.49)$$

$$N = -(2n_a + n_b + n_c) = 2i, \quad (9.50)$$

$$2a_0 = k_1 \cdot D \cdot k_1 + k_2 \cdot D \cdot k_2, \quad (9.51)$$

$$F_N = \left(1 + \sqrt{\frac{1 - \cos \phi}{1 + \cos \phi}}\right)^N + \left(1 - \sqrt{\frac{1 - \cos \phi}{1 + \cos \phi}}\right)^N. \quad (9.52)$$

The total high energy scattering amplitude can then be calculated to be

$$\begin{aligned} A_{D-Par} &= A_{D-Par}^{(0 \rightarrow 1)} + A_{D-Par}^{(1 \rightarrow \infty)} \\ &\simeq (-1)^n 4 (2i)^{2a_0} (2E \sin \phi)^{2n} \left(-\frac{1}{8M \sin^2 \phi}\right)^{q+q'} \\ &\cdot \sum_{i=0}^{q+q'} \sum_{j=0}^i \binom{q+q'}{i} \binom{i}{j} (-2)^i (1 - \cos \phi)^j (1 + \cos \phi)^{i-j} \cdot 2^{-2a-2-N} B\left(a+1, \frac{b+1}{2}\right) F_N. \end{aligned} \quad (9.53)$$

The high energy limit of the beta function is

$$B\left(a+1, \frac{b+1}{2}\right) \simeq B\left(a_0+1, \frac{b_0+1}{2}\right) \frac{a_0^{n_a} \left(\frac{b_0}{2}\right)^{n_b/2}}{\left(a_0 + \frac{b_0}{2}\right)^{n_a+n_b/2}}. \quad (9.54)$$

Finally we get the high energy scattering amplitudes at mass level $M^2 = 2(n-1)$

$$\begin{aligned} A_{D-Par} &= A_{D-Par}^{(0 \rightarrow 1)} + A_{D-Par}^{(1 \rightarrow \infty)} \\ &= (-1)^{a_0} E^{2n} \left(-\frac{1}{2M}\right)^{q+q'} B\left(a_0+1, \frac{b_0+1}{2}\right) \\ &\cdot \sum_{i=0}^{q+q'} \sum_{j=0}^i \binom{q+q'}{i} \binom{i}{j} (-2)^{-i} (1 + \cos \phi)^i (-1)^j F_N \\ &= 2(-1)^{a_0} E^{2n} \left(-\frac{1}{2M}\right)^{q+q'} B\left(a_0+1, \frac{b_0+1}{2}\right) \\ &== 2(-1)^{a_0} E^{2n} \left(-\frac{1}{2M}\right)^{q+q'} \frac{\Gamma(a_0+1) \Gamma(\frac{b_0+1}{2})}{\Gamma(a_0 + \frac{b_0}{2} + \frac{3}{2})} \end{aligned} \quad (9.55)$$

where the high energy limit of $B(a_0 + 1, \frac{b_0+1}{2})$ is independent of $q + q'$. We thus have explicitly shown that there is only one independent high energy scattering amplitude at each fixed mass level. It is a remarkable result that the ratios $\left(-\frac{1}{2M}\right)^{q+q'}$ for different high energy scattering amplitudes at each fixed mass level is consistent with Eq.(5.60) for the scattering without D-brane as expected. In general, for an incoming tensor state and an outgoing tensor state scatterings, the ratios are $\Sigma_{i=1}^2 \left(-\frac{1}{2M_i}\right)^{(q_i+q'_i)}$.

Finally, one notes that the exponential fall-off behavior in energy E is hidden in the high energy beta function. Since the arguments of $\Gamma(a_0 + 1)$ and $\Gamma(a_0 + \frac{b_0}{2} + \frac{3}{2})$ in Eq.(9.55) are negative in the high-energy limit, one needs to use the well known formula

$$\Gamma(x) = \frac{\pi}{\sin(\pi x) \Gamma(1-x)} \quad (9.56)$$

to calculate the large negative x expansion of these Γ functions, and obtain the Regge-pole structure [35] of the amplitude. This is to be compared with the power-law behavior with Regge-pole structure for the D-domain-wall scattering to be discussed in the next section.

4. Hard strings scattered from D-domain-wall

We have shown, in the last section, that the linear relations for string/string scatterings persist for the string/D p -brane scatterings with $p \geq 0$. In particular, the linear relations for the D-particle scatterings [41] were explicitly demonstrated. All the high energy string/D p -brane scattering amplitudes with $p \geq 0$ behave as exponential fall-off as was claimed in [52, 135–139]. In this section, in contrast to the common wisdom, we show that [45], instead of the exponential fall-off behavior of the form factors with Regge-pole structure, the high energy scattering amplitudes of string scattered from D24-brane, or Domain-wall, behave as *power-law* with Regge-pole structure. This is to be compared with the well-known power-law form factors without Regge-pole structure of the D-instanton scatterings to be discussed in Eq.(9.74) below.

This discovery makes Domain-wall scatterings an unique example of a hybrid of string and field theory scatterings. Our calculation will be done for bosonic string scatterings of arbitrary massive string states from D24-brane. Moreover, we discover that the usual linear relations [41] of high energy string scattering amplitudes at each fixed mass level, Eq.(9.55), breaks down for the Domain-wall scatterings [45]. This result gives a strong evidence that

the existence of the infinite linear relations, or stringy symmetries, of high energy string scattering amplitudes is responsible for the softer, exponential fall-off high energy string scatterings than the power-law field theory scatterings.

We consider an incoming tachyon closed string state with momentum k_1 and an angle of incidence ϕ and an outgoing massive closed string state $(\alpha_{-1}^T)^{n-2q} (\alpha_{-2}^L)^q \otimes (\tilde{\alpha}_{-1}^T)^{n-2q'} (\tilde{\alpha}_{-2}^L)^{q'} |0\rangle$ with momentum k_2 and an angle of reflection θ . The kinematic setup is

$$e^P = \frac{1}{M} (-E, k_2 \cos \theta, -k_2 \sin \theta) = \frac{k_2}{M}, \quad (9.57)$$

$$e^L = \frac{1}{M} (-k_2, E \cos \theta, -E \sin \theta), \quad (9.58)$$

$$e^T = (0, \sin \theta, \cos \theta), \quad (9.59)$$

$$k_1 = (E, -k_1 \cos \phi, -k_1 \sin \phi), \quad (9.60)$$

$$k_2 = (-E, k_2 \cos \theta, -k_2 \sin \theta). \quad (9.61)$$

In the high energy limit, the angle of incidence ϕ is identified to the angle of reflection θ , and e^P approaches e^L , $k_1, k_2 \simeq E$. For the case of Domain-wall scattering $Diag D_{\mu\nu} = (-1, 1, -1)$, and we have

$$a_0 \equiv k_1 \cdot D \cdot k_1 \sim -2E^2 \sin^2 \phi - 2M_1^2 \cos^2 \phi + M_1^2, \quad (9.62)$$

$$\begin{aligned} b_0 &\equiv 2k_1 \cdot k_2 + 1 \\ &\sim 4E^2 \sin^2 \phi + 4M_1^2 \cos^2 \phi - (M_1^2 + M^2) + 1, \end{aligned} \quad (9.63)$$

The scattering amplitude can be calculated to be

$$\begin{aligned} A_{D-Wall} &= (-1)^{q+q'} \int d^2 z_1 d^2 z_2 (z_1 - \bar{z}_1)^{k_1 \cdot D \cdot k_1} (z_2 - \bar{z}_2)^{k_2 \cdot D \cdot k_2} |z_1 - z_2|^{2k_1 \cdot k_2} |z_1 - \bar{z}_2|^{2k_1 \cdot D \cdot k_2} \\ &\cdot \left[\frac{ie^T \cdot k_1}{z_1 - z_2} + \frac{ie^T \cdot D \cdot k_1}{\bar{z}_1 - z_2} + \frac{ie^T \cdot D \cdot k_2}{\bar{z}_2 - z_2} \right]^{n-2q} \\ &\cdot \left[\frac{ie^T \cdot D \cdot k_1}{z_1 - \bar{z}_2} + \frac{ie^T \cdot k_1}{\bar{z}_1 - \bar{z}_2} + \frac{ie^T \cdot D \cdot k_2}{z_2 - \bar{z}_2} \right]^{n-2q'} \\ &\cdot \left[\frac{e^L \cdot k_1}{(z_1 - z_2)^2} + \frac{e^L \cdot D \cdot k_1}{(\bar{z}_1 - z_2)^2} + \frac{e^L \cdot D \cdot k_2}{(\bar{z}_2 - z_2)^2} \right]^q \left[\frac{e^L \cdot D \cdot k_1}{(z_1 - \bar{z}_2)^2} + \frac{e^L \cdot k_1}{(\bar{z}_1 - \bar{z}_2)^2} + \frac{e^L \cdot D \cdot k_2}{(z_2 - \bar{z}_2)^2} \right]^{q'}. \end{aligned} \quad (9.64)$$

Set $z_1 = iy$ and $z_2 = i$ to fix the $SL(2, R)$ gauge, and include the Jacobian $d^2 z_1 d^2 z_2 \rightarrow 4(1 - y^2) dy$, we have, for the $(0 \rightarrow 1)$ channel,

$$\begin{aligned}
& A_{D-Wall}^{(0 \rightarrow 1)} \\
& \simeq 4 (2i)^{k_1 \cdot D \cdot k_1 + k_2 \cdot D \cdot k_2} \left(\frac{E \sin 2\phi}{2} \right)^{2n} \left(\frac{1}{2M \cos^2 \phi} \right)^{q+q'} \\
& \cdot \sum_{i=0}^{q+q'} \binom{q+q'}{i} 2^i \int_0^1 dy y^{k_2 \cdot D \cdot k_2} (1-y)^{2k_1 \cdot k_2 + 1} \\
& \cdot (1+y)^{2k_1 \cdot D \cdot k_2 + 1} \left[\frac{1+y}{1-y} \right]^{2n-(q+q')} \left(\frac{1}{1-y} \right)^i \\
& \simeq \left(\frac{E \sin 2\phi}{2} \right)^{2n} \left(\frac{1}{2M \cos^2 \phi} \right)^{q+q'} \\
& \cdot \sum_{i=0}^{q+q'} \binom{q+q'}{i} \cdot B \left(a_0 + 1, \frac{b+1}{2} \right) F_i
\end{aligned} \tag{9.65}$$

where

$$b = b_0 + n_b = b_0 - 2n + (q + q') - i, \tag{9.66}$$

$$F_i \equiv \left(1 + \sqrt{\left| \frac{b}{2a_0 + b} \right|} \right)^i + \left(1 - \sqrt{\left| \frac{b}{2a_0 + b} \right|} \right)^i \tag{9.67}$$

$$\simeq \left[(1 + 2C_i E \sin \phi)^i + (1 - 2C_i E \sin \phi)^i \right] \tag{9.68}$$

with

$$C_i \equiv \sqrt{\left| \frac{1}{M_1^2 - M^2 + 1 - 2n + (q + q') - i} \right|}. \tag{9.69}$$

F_i in Eq.(9.67) is the high energy limit of the generalized hypergeometric function ${}_3F_2 \left(\frac{b+1}{2}, -\left[\frac{i}{2}\right], \frac{1}{2} - \left[\frac{i}{2}\right]; a_0 + \frac{b+3}{2}, \frac{1}{2}; 1 \right)$ [41].

At this stage, it is crucial to note that

$$b \simeq b_0 \simeq -2a_0 \tag{9.70}$$

in the high energy limit for the Domain-wall scatterings. As a result, F_i reduces to the form of Eq.(9.68), and depends on the energy E . Thus in contrast to the generic Dp-brane scatterings with $p \geq 0$, which contain two independent kinematic variables, there is only *one* kinematic variable for the special case of Domain-wall scatterings. It thus becomes meaningless to study high energy, fixed angle scattering process for the Domain-wall scatterings. As we

will see in the following calculation, this peculiar property will reduce the high energy beta function in Eq.(9.65) from exponential to power-law behavior and, simultaneously, breaks down the linear relations as we had in Eq.(9.55) for the D-particle scatterings.

Finally, the scattering amplitude for the $(0 \rightarrow 1)$ channel can be calculated to be (similar result can be obtained for the $(1 \rightarrow \infty)$ channel)

$$\begin{aligned}
& A_{D-Wall}^{(0 \rightarrow 1)} \\
& \simeq \left(\frac{E \sin 2\phi}{2} \right)^{2n} \left(\frac{1}{2M \cos^2 \phi} \right)^{q+q'} B \left(a_0 + 1, \frac{b_0 + 1}{2} \right) \\
& \cdot \sum_{i=0}^{q+q'} \binom{q+q'}{i} \cdot \frac{(a_0)_0 \left(\frac{b_0}{2} \right)_{n_b/2}}{\left(a_0 + \frac{b_0}{2} \right)_{n_b/2}} (1 + 2C_i E \sin \phi)^i
\end{aligned} \tag{9.71}$$

$$\begin{aligned}
& \simeq \left(\frac{\cos \phi}{\sqrt{2}} \right)^{2n} \left(\frac{E \sin \phi}{M \sqrt{|M_1^2 - 2M^2 - 1|} \cos^2 \phi} \right)^{q+q'} \\
& \cdot \frac{\Gamma(a_0 + 1) \Gamma(\frac{b_0+1}{2})}{\Gamma(a_0 + \frac{b_0}{2} + \frac{3}{2})} \frac{1}{\left(\frac{M_1^2 - M^2 + 1}{2} \right)_{-n}}
\end{aligned} \tag{9.72}$$

where $(\alpha)_n \equiv \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}$ for integer n . On the other hand, since the argument of $\Gamma(a_0 + 1)$ in Eq.(9.72) is negative in the high energy limit, we have, by using Eq.(9.13) and Eq.(9.70),

$$\begin{aligned}
& \frac{\Gamma(a_0 + 1) \Gamma(\frac{b_0+1}{2})}{\Gamma(a_0 + \frac{b_0}{2} + \frac{3}{2})} \simeq \frac{\pi}{\sin(\pi a_0) \Gamma(-a_0)} \frac{\Gamma(\frac{b_0+1}{2})}{\Gamma(\frac{M_1^2 - M^2 + 1}{2})} \\
& \sim \frac{1}{\sin(\pi a_0)} \frac{1}{(E \sin \phi)^{2(n-1)}}.
\end{aligned} \tag{9.73}$$

Note that the $\sin(\pi a_0)$ factor in the denominator of Eq.(9.73) gives the Regge-pole structure, and the energy dependence $E^{-2(n-1)}$ gives the power-law behavior in the high energy limit. As a result, the scattering amplitude for the Domain-wall in Eq.(9.72) behaves like *power-law with the Regge-pole structure*.

The crucial differences between the Domain-wall scatterings in Eq.(9.72) and the D-particle scatterings (or any other Dp-brane scatterings except Domain-wall and D-instanton scatterings) in Eq.(9.55) is the kinematic relation Eq.(9.70). For the case of D-particle scatterings [41], the corresponding factors for both F_i in Eq.(9.67) and the fraction in Eq.(9.71) are independent of energy in the high energy limit, and, as a result, the amplitudes contain no $q + q'$ dependent energy power factor. So one gets the high energy linear relations for the

D-particle scattering amplitudes. On the contrary, for the case of Domain-wall scatterings, both F_i in Eq.(9.68) and the fraction in Eq.(9.71) depend on energy due to the condition Eq.(9.70). The summation in Eq.(9.71) is then dominated by the term $i = q + q'$, and the whole scattering amplitude Eq.(9.72) contains a $q + q'$ dependent energy power factor. As a result, the usual linear relations for the high energy scattering amplitudes break down for the Domain-wall scatterings.

It is crucial to note that the mechanism, Eq.(9.70), to drive the exponential fall-off form factor of the D-particle scatterings to the power-law one of the Domain-wall scatterings is exactly the same as the mechanism to break down the expected linear relations for the domain-wall scatterings in the high energy limit. In conclusion, this result gives a strong evidence that the existence of the infinite linear relations, or stringy symmetries, of high energy string scattering amplitudes is responsible for the softer, exponential fall-off high energy string scatterings than the power-law field theory scatterings.

Another interesting case of D-brane scatterings is the massless form factor of scatterings of D-instanton [52, 135–139]

$$\frac{\Gamma(s)\Gamma(t)}{\Gamma(s+t+1)} \rightarrow \frac{1}{st}, \text{ as } s \rightarrow 0, \quad (9.74)$$

which contains no Regge-pole structure. In Eq.(9.74), s, t are the Mandelstam variables. Eq.(9.74) can be easily generalized to the scatterings of arbitrary massive string states in the high energy limit. To compare the D-instanton scatterings with the Domain-wall scatterings in Eq.(9.73), one notes that in both cases there is only *one* kinematic variable and, as a result, behave as power-law at high energies [126].

On the other hand, since t is large negative in the high energy limit [35], the application of Eq.(9.13) to Eq.(9.74) produces no $\sin(\pi a_0)$ factor in contrast to the Domain-wall scatterings. So there is *no Regge-pole structure* for the D-instanton scatterings. We conclude that the very condition of Eq.(9.70) makes Domain-wall scatterings an unique example of a hybrid of string and field theory scatterings.

5. A brief review of ${}_2F_1$ and ${}_3F_2$

In this section, we review the definitions and some formulas of hypergeometric function ${}_2F_1$ and generalized hypergeometric function ${}_3F_2$ which we used in the text. hypergeometric

functions form an important class of special functions. Many elementary special functions are special cases of ${}_2F_1$. The hypergeometric function ${}_2F_1$ is defined to be (α, β, γ constant)

$$\begin{aligned} {}_2F_1(\alpha, \beta, \gamma; x) &= 1 + \frac{\alpha\beta}{\gamma} \frac{x}{1!} + \frac{\alpha(\alpha+1)\beta(\beta+1)}{\gamma(\gamma+1)} \frac{x^2}{2!} + \cdots = \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{(\gamma)_n} \frac{x^n}{n!} \\ &= \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)} \sum_{n=0}^{\infty} \frac{\Gamma(\alpha+n)\Gamma(\beta+n)}{\Gamma(\gamma+n)} \frac{x^n}{n!} \end{aligned} \quad (9.75)$$

where

$$(\alpha)_0 = 1, (\alpha)_n = \alpha(\alpha+1)(\alpha+2) \cdots (\alpha+n-1) = \frac{(\alpha+n-1)!}{(\alpha-1)!}. \quad (9.76)$$

The hypergeometric function ${}_2F_1$ is a solution, at the singular point $x = 0$ with indicial root $r = 0$, of the Gauss's hypergeometric differential equation

$$x(1-x)u'' + [\gamma - (\alpha + \beta + 1)]u' - \alpha\beta u = 0, \quad (9.77)$$

which contains three regular singularities $x = 0, 1, \infty$. The second solution of Eq. (9.77) with indicial root $r = 1 - \gamma$ can be expressed in terms of ${}_2F_1$ as following ($\gamma \neq \text{integer}$)

$$u_2(x) = x^{1-\gamma} {}_2F_1(\alpha - \gamma + 1, \beta - \gamma + 1, 2 - \gamma, x). \quad (9.78)$$

Other solutions of Eq. (9.77), which corresponds to singularities $x = 1, \infty$, can also be expressed in terms of the hypergeometric function ${}_2F_1$. The following identity

$${}_2F_1(\alpha, \beta, \gamma; x) = (1-x)^{\gamma-\alpha-\beta} {}_2F_1(\gamma-\alpha, \gamma-\beta, \gamma; x), \quad (9.79)$$

which we used in the text can then be derived.

${}_2F_1$ has an integral representation

$${}_2F_1(\alpha, \beta, \gamma; x) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \int_0^1 dy y^{\beta-1} (1-y)^{\gamma-\beta-1} (1-yx)^{-\alpha}, \quad (9.80)$$

which can be used to do analytic continuation. Eq.(9.80) with $x = -1$ was repeatedly used in the text in our calculations of string scattering amplitudes with D-brane.

There exists interesting relations among hypergeometric function ${}_2F_1$ with different arguments

$$x^{-p}(1-x)^{-q} {}_2F_1(\alpha, \beta, \gamma; x) = t^{-p'}(1-t)^{-q'} {}_2F_1(\alpha', \beta', \gamma'; t), \quad (9.81)$$

where $x = \varphi(t)$ is an algebraic function with degree up to six. As an example, the quadratic transformation formula

$${}_2F_1(\alpha, \beta, 1 + \alpha - \beta; x) = (1 - x)^{-\alpha} {}_2F_1\left(\frac{\alpha}{2}, \frac{1 + \alpha - 2\beta}{2}, 1 + \alpha - \beta; \frac{-4x}{(1 - x)^2}\right), \quad (9.82)$$

can be used to derive the Kummer's relation

$${}_2F_1(\alpha, \beta, 1 + \alpha - \beta, -1) = \frac{\Gamma(1 + \alpha - \beta)\Gamma(1 + \frac{\alpha}{2})}{\Gamma(1 + \alpha)\Gamma(1 + \frac{\alpha}{2} - \beta)}, \quad (9.83)$$

which is crucial to reduce the scattering amplitudes of string from D-brane to the usual beta function.

In summing up the $(0 \rightarrow 1)$ and $(1 \rightarrow \infty)$ channel scattering amplitudes, we have used the master formula

$$\begin{aligned} I &= I_{(0 \rightarrow 1)} + I_{(1 \rightarrow \infty)} \\ &= 2^{-2a-1-N} \int_0^1 dt \, t^b (1-t)^a (1+t)^a \left[(1+t)^N + (1-t)^N \right] \\ &= 2^{-2a-1-N} \sum_{n=0}^{\left[\frac{N}{2}\right]} \binom{N}{2n} \cdot B\left(a+1, \frac{b+1}{2} + n\right) \\ &= 2^{-2a-1-N} \cdot B\left(a+1, \frac{b+1}{2}\right) \cdot {}_3F_2\left(\frac{b+1}{2}, -\left[\frac{N}{2}\right], \frac{1}{2} - \left[\frac{N}{2}\right]; a + \frac{b+3}{2}, \frac{1}{2}; 1\right) \end{aligned} \quad (9.84)$$

In Eq.(9.84), B is the beta function and ${}_3F_2$ is the generalized hypergeometric function, which is defined to be

$${}_3F_2(\alpha_1, \alpha_2, \alpha_3; \gamma_1, \gamma_2; x) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n (\alpha_2)_n (\alpha_3)_n}{(\gamma_1)_n (\gamma_2)_n} \frac{x^n}{n!}. \quad (9.85)$$

For those arguments of ${}_3F_2$ in Eq. (9.84), the series of the generalized hypergeometric

function ${}_3F_2$ terminates to a finite sum. For example,

$$\begin{aligned}
N = 0 : {}_3F_2 &= 1, \\
N = 1 : {}_3F_2 &= 1, \\
N = 2 : {}_3F_2 &= \frac{a+b+2}{a+\frac{b+3}{2}}, \\
N = 3 : {}_3F_2 &= \frac{a+2b+3}{a+\frac{b+3}{2}}, \\
N = 4 : {}_3F_2 &= \frac{a^2+4ab+2b^2+7a+12b+12}{\left(a+\frac{b+3}{2}\right)\left(a+\frac{b+5}{2}\right)}, \\
N = 5 : {}_3F_2 &= \frac{a^2+6ab+9b^2+4a+22b+20}{\left(a+\frac{b+3}{2}\right)\left(a+\frac{b+5}{2}\right)}. \tag{9.86}
\end{aligned}$$

B. Scatterings from O-planes

Being a consistent theory of quantum gravity, string theory is remarkable for its soft ultraviolet structure. This is mainly due to two closely related fundamental characteristics of high-energy string scattering amplitudes. The first is the softer exponential fall-off behavior of the form factors of high-energy string scatterings in contrast to the power-law field theory scatterings. The second is the existence of infinite Regge poles in the form factor of string scattering amplitudes. The existence of infinite linear relations discussed in part II of the review constitutes the *third* fundamental characteristics of high energy string scatterings.

In the previous section, we showed that these linear relations persist [41] for string scattered from generic Dp -brane [135] except D-instanton and D-domain-wall. For the scattering of D-instanton, the form factor exhibits the well-known power-law behavior without Regge pole structure, and thus resembles a field theory amplitude. For the special case of D-domain-wall scattering [52], it was discovered [45] that its form factor behaves as *power-law* with infinite *open* Regge pole structure at high energies. This discovery makes D-domain-wall scatterings an unique example of a hybrid of string and field theory scatterings.

Moreover, it was shown [45] that the linear relations break down for the D-domain-wall scattering due to this unusual power-law behavior. This result seems to imply the coexistence of linear relations and soft UV structure of string scatterings. In order to further uncover the mysterious relations among these three fundamental characteristics of string scatterings, namely, the soft UV structure, the existence of infinite Regge poles and the newly discovered

linear relations stated above, it will be important to study more string scatterings, which exhibit the unusual behaviors in the high energy limit.

In this section, we calculate massive closed string states at arbitrary mass levels scattered from Orientifold planes in the high energy, fixed angle limit [46]. The scatterings of massless states from Orientifold planes were calculated previously by using the boundary states formalism [47–50], and on the worldsheet of real projected plane RP_2 [51]. Many speculations were made about the scatterings of *massive* string states, in particular, for the case of O-domain-wall scatterings. It is one of the purposes of this section to clarify these speculations and to discuss their relations with the three fundamental characteristics of high energy string scatterings.

For the generic Op -planes with $p \geq 0$, one expects to get the infinite linear relations except O-domain-wall scatterings. For simplicity, we consider only the case of O-particle scatterings [46]. For the case of O-particle scatterings, we will obtain infinite linear relations among high energy scattering amplitudes of different string states. We also confirm that there exist only t -channel closed string Regge poles in the form factor of the O-particle scatterings amplitudes as expected.

For the case of O-domain-wall scatterings, we find that, like the well-known D-instanton scatterings, the amplitudes behave like field theory scatterings, namely *UV power-law without Regge pole*. In addition, we will show that there exist only finite number of t -channel closed string poles in the form factor of O-domain-wall scatterings, and the masses of the poles are bounded by the masses of the external legs [46]. We thus confirm that all massive closed string states do couple to the O-domain-wall as was conjectured previously [51, 52]. This is also consistent with the boundary state descriptions of O-planes.

For both cases of O-particle and O-domain-wall scatterings, we confirm that there exist no s -channel open string Regge poles in the form factor of the amplitudes as O-planes were known to be not dynamical. However, the usual claim that there is a thickness of order $\sqrt{\alpha'}$ for the O-domain-wall is misleading as the UV behavior of its scatterings is power-law instead of exponential fall-off.

1. Hard strings scattered from O-particle

We use the real projected plane RP_2 as the worldsheet diagram for the scatterings of Orientifold planes [46]. The standard propagators of the left and right moving fields are

$$\langle X^\mu(z) X^\nu(w) \rangle = -\eta^{\mu\nu} \log(z-w), \quad (9.87)$$

$$\langle \tilde{X}^\mu(\bar{z}) \tilde{X}^\nu(\bar{w}) \rangle = -\eta^{\mu\nu} \log(\bar{z}-\bar{w}). \quad (9.88)$$

In addition, there are also nontrivial correlator between the right and left moving fields as well

$$\langle X^\mu(z) \tilde{X}^\nu(\bar{w}) \rangle = -D^{\mu\nu} \ln(1+z\bar{w}). \quad (9.89)$$

As in the usual convention [52, 135–139], the matrix D reverses the sign for fields satisfying Dirichlet boundary condition. The wave functions of a tensor at general mass level can be written as

$$T_{\mu_1 \dots \mu_n} = \frac{1}{2} \left[\varepsilon_{\mu_1 \dots \mu_n} e^{ik \cdot x} + (D \cdot \varepsilon)_{\mu_1} \dots (D \cdot \varepsilon)_{\mu_n} e^{iD \cdot k \cdot x} \right] \quad (9.90)$$

where

$$\varepsilon_{\mu_1 \dots \mu_n} \equiv \varepsilon_{\mu_1} \dots \varepsilon_{\mu_n}. \quad (9.91)$$

The vertex operators corresponding to the above wave functions are

$$V(\varepsilon, k, z, \bar{z}) = \frac{1}{2} \left[\varepsilon_{\mu_1 \dots \mu_n} V^{\mu_1 \dots \mu_n}(k, z, \bar{z}) + (D \cdot \varepsilon)_{\mu_1} \dots (D \cdot \varepsilon)_{\mu_n} V^{\mu_1 \dots \mu_n}(D \cdot k, z, \bar{z}) \right]. \quad (9.92)$$

For simplicity, we are going to calculate one tachyon and one massive closed string state scattered from the O-particle in the high energy limit. One expects to get similar results for the generic Op -plane scatterings with $p \geq 0$ except O-domain-wall scatterings, which will be discussed in the next section. For this case $D_{\mu\nu} = -\delta_{\mu\nu}$, and the kinematic setup are

$$e^P = \frac{1}{M} (-E, -k_2, 0) = \frac{k_2}{M}, \quad (9.93)$$

$$e^L = \frac{1}{M} (-k_2, -E, 0), \quad (9.94)$$

$$e^T = (0, 0, 1), \quad (9.95)$$

$$k_1 = (E, k_1 \cos \phi, -k_1 \sin \phi), \quad (9.96)$$

$$k_2 = (-E, -k_2, 0) \quad (9.97)$$

where e^P , e^L and e^T are polarization vectors of the tensor state k_2 on the high energy scattering plane. One can easily calculate the following kinematic relations in the high energy limit

$$e^T \cdot k_2 = e^L \cdot k_2 = 0, \quad (9.98)$$

$$e^T \cdot k_1 = -k_1 \sin \phi \sim -E \sin \phi, \quad (9.99)$$

$$e^T \cdot D \cdot k_1 = k_1 \sin \phi \sim E \sin \phi, \quad (9.100)$$

$$e^T \cdot D \cdot k_2 = 0, \quad (9.101)$$

$$e^L \cdot k_1 = \frac{1}{M} [k_2 E - k_1 E \cos \phi] \sim \frac{E^2}{M} (1 - \cos \phi), \quad (9.102)$$

$$e^L \cdot D \cdot k_1 = \frac{1}{M} [k_2 E + k_1 E \cos \phi] \sim \frac{E^2}{M} (1 + \cos \phi), \quad (9.103)$$

$$e^L \cdot D \cdot k_2 = \frac{1}{M} [-k_2 E - k_2 E] \sim -\frac{2E^2}{M}. \quad (9.104)$$

We define

$$a_0 \equiv k_1 \cdot D \cdot k_1 = -E^2 - k_1^2 \sim -2E^2, \quad (9.105)$$

$$a'_0 \equiv k_2 \cdot D \cdot k_2 = -E^2 - k_2^2 \sim -2E^2, \quad (9.106)$$

$$b_0 \equiv k_1 \cdot k_2 = (E^2 - k_1 k_2 \cos \phi) \sim E^2 (1 - \cos \phi), \quad (9.107)$$

$$c_0 \equiv k_1 \cdot D \cdot k_2 = (E^2 + k_1 k_2 \cos \phi) \sim E^2 (1 + \cos \phi), \quad (9.108)$$

and the Mandelstam variables can be calculated to be

$$t \equiv -(k_1 + k_2)^2 = M_1^2 + M_2^2 - 2k_1 \cdot k_2 = M_2^2 - 2(1 + b_0), \quad (9.109)$$

$$s \equiv \frac{1}{2} k_1 \cdot D \cdot k_1 = \frac{1}{2} a_0, \quad (9.110)$$

$$u = -2k_1 \cdot D \cdot k_2 = -2c_0. \quad (9.111)$$

In the high energy limit, we will consider an incoming tachyon state k_1 and an outgoing tensor state k_2 of the following form

$$(\alpha_{-1}^T)^{n-2q} (\alpha_{-2}^L)^q \otimes (\tilde{\alpha}_{-1}^T)^{n-2q'} (\tilde{\alpha}_{-2}^L)^{q'} |0\rangle. \quad (9.112)$$

For simplicity, we have omitted above a possible high energy vertex $(\alpha_{-1}^L)^r \otimes (\tilde{\alpha}_{-1}^L)^{r'}$ [41, 54].

For this case, with momentum conservation on the O-planes, we have

$$a_0 + b_0 + c_0 = M_1^2 = -2. \quad (9.113)$$

The high energy scattering amplitude can then be written as

$$\begin{aligned}
A^{RP_2} &= \int d^2 z_1 d^2 z_2 \frac{1}{2} \left[V(k_1, z_1) \tilde{V}(k_1, \bar{z}_1) + V(D \cdot k_1, z_1) \tilde{V}(D \cdot k_1, \bar{z}_1) \right] \\
&\quad \cdot \frac{1}{2} \varepsilon_{T^{n-2q} L^q, T^{n-2q'} L^{q'}} V^{T^{n-2q} L^q}(k_2, z_2) \tilde{V}^{T^{n-2q'} L^{q'}}(k_2, \bar{z}_2) \\
&\quad + (D \cdot \varepsilon_T)^{n-2q} (D \cdot \varepsilon_L)^q (D \cdot \tilde{\varepsilon}_T)^{n-2q'} (D \cdot \tilde{\varepsilon}_L)^{q'} V^{T^{n-2q} L^q}(D \cdot k_2, z_2) \\
&\quad \cdot \tilde{V}^{T^{n-2q'} L^{q'}}(D \cdot k_2, \bar{z}_2) \\
&= A_1 + A_2 + A_3 + A_4
\end{aligned}$$

where

$$\begin{aligned}
A_1 &= \frac{1}{4} \varepsilon_{T^{n-2q} L^q, T^{n-2q'} L^{q'}} \int d^2 z_1 d^2 z_2 \\
&\quad \cdot \left\langle V(k_1, z_1) \tilde{V}(k_1, \bar{z}_1) V^{T^{n-2q} L^q}(k_2, z_2) \tilde{V}^{T^{n-2q'} L^{q'}}(k_2, \bar{z}_2) \right\rangle, \tag{9.114}
\end{aligned}$$

$$\begin{aligned}
A_2 &= \frac{1}{4} \varepsilon_{T^{n-2q} L^q, T^{n-2q'} L^{q'}} \int d^2 z_1 d^2 z_2 \\
&\quad \cdot \left\langle V(D \cdot k_1, z_1) \tilde{V}(D \cdot k_1, \bar{z}_1) V^{T^{n-2q} L^q}(k_2, z_2) \tilde{V}^{T^{n-2q'} L^{q'}}(k_2, \bar{z}_2) \right\rangle, \tag{9.115}
\end{aligned}$$

$$\begin{aligned}
A_3 &= \frac{1}{4} (D \cdot \varepsilon_T)^{n-2q} (D \cdot \varepsilon_L)^q (D \cdot \tilde{\varepsilon}_T)^{n-2q'} (D \cdot \tilde{\varepsilon}_L)^{q'} \\
&\quad \cdot \int d^2 z_1 d^2 z_2 \left\langle V(k_1, z_1) \tilde{V}(k_1, \bar{z}_1) V^{T^{n-2q} L^q}(D \cdot k_2, z_2) \tilde{V}^{T^{n-2q'} L^{q'}}(D \cdot k_2, \bar{z}_2) \right\rangle, \tag{9.116}
\end{aligned}$$

$$\begin{aligned}
A_4 &= \frac{1}{4} (D \cdot \varepsilon_T)^{n-2q} (D \cdot \varepsilon_L)^q (D \cdot \tilde{\varepsilon}_T)^{n-2q'} (D \cdot \tilde{\varepsilon}_L)^{q'} \\
&\quad \cdot \int d^2 z_1 d^2 z_2 \left\langle V(D \cdot k_1, z_1) \tilde{V}(D \cdot k_1, \bar{z}_1) V^{T^{n-2q} L^q}(D \cdot k_2, z_2) \tilde{V}^{T^{n-2q'} L^{q'}}(D \cdot k_2, \bar{z}_2) \right\rangle. \tag{9.117}
\end{aligned}$$

One can easily see that

$$A_1 = A_4, A_2 = A_3. \tag{9.118}$$

We will choose to calculate A_1 and A_2 . For the case of A_1 , we have

$$\begin{aligned}
4A_1 &= \varepsilon_{T^{n-2q}L^q, T^{n-2q'}L^{q'}} \int d^2z_1 d^2z_2 \cdot \\
&\left\langle e^{ik_1X}(z_1) e^{ik_1\tilde{X}}(\bar{z}_1) (\partial X^T)^{n-2q} (i\partial^2 X^L)^q e^{ik_2X}(z_2) (\bar{\partial}\tilde{X}^T)^{n-2q'} (i\bar{\partial}^2\tilde{X}^L)^{q'} e^{ik_2\tilde{X}}(\bar{z}_2) \right\rangle \\
&= (-1)^{q+q'} \int d^2z_1 d^2z_2 (1+z_1\bar{z}_1)^{a_0} (1+z_2\bar{z}_2)^{a'_0} |z_1-z_2|^{2b_0} |1+z_1\bar{z}_2|^{2c_0} \\
&\cdot \left[\frac{ie^T \cdot k_1}{z_1-z_2} - \frac{ie^T \cdot D \cdot k_1}{1+\bar{z}_1 z_2} \bar{z}_1 - \frac{ie^T \cdot D \cdot k_2}{1+\bar{z}_2 z_2} \bar{z}_2 \right]^{n-2q} \\
&\cdot \left[-\frac{ie^T \cdot D \cdot k_1}{1+z_1\bar{z}_2} z_1 + \frac{ie^T \cdot k_1}{\bar{z}_1-\bar{z}_2} - \frac{ie^T \cdot D \cdot k_2}{1+z_2\bar{z}_2} z_2 \right]^{n-2q'} \\
&\cdot \left[\frac{e^L \cdot k_1}{(z_1-z_2)^2} + \frac{e^L \cdot D \cdot k_1}{(1+\bar{z}_1 z_2)^2} \bar{z}_1^2 + \frac{e^L \cdot D \cdot k_2}{(1+\bar{z}_2 z_2)^2} \bar{z}_2^2 \right]^q \\
&\cdot \left[\frac{e^L \cdot D \cdot k_1}{(1+z_1\bar{z}_2)^2} z_1^2 + \frac{e^L \cdot k_1}{(\bar{z}_1-\bar{z}_2)^2} + \frac{e^L \cdot D \cdot k_2}{(1+z_2\bar{z}_2)^2} z_2^2 \right]^{q'}. \tag{9.119}
\end{aligned}$$

To fix the modulus group on RP_2 , choosing $z_1 = r$ and $z_2 = 0$ and we have

$$\begin{aligned}
4A_1 &= (-1)^n \int_0^1 dr^2 (1+r^2)^{a_0} r^{2b_0} \\
&\cdot \left[\frac{e^T \cdot k_1}{r} - \frac{e^T \cdot D \cdot k_1}{1} r \right]^{n-2q} \cdot \left[-\frac{e^T \cdot D \cdot k_1}{1} r + \frac{e^T \cdot k_1}{r} \right]^{n-2q'} \\
&\cdot \left[\frac{e^L \cdot k_1}{r^2} + \frac{e^L \cdot D \cdot k_1}{1} r^2 \right]^q \cdot \left[\frac{e^L \cdot D \cdot k_1}{1} r^2 + \frac{e^L \cdot k_1}{r^2} \right]^{q'} \\
&= (-1)^n (E \sin \phi)^{2n} \left(\frac{2 \cos^2 \frac{\phi}{2}}{M \sin^2 \phi} \right)^{q+q'} \sum_{i=0}^{q+q'} \binom{q+q'}{i} \left(\frac{\sin^2 \frac{\phi}{2}}{\cos^2 \frac{\phi}{2}} \right)^i \\
&\cdot \int_0^1 dr^2 (1+r^2)^{a_0+2n-2(q+q')} \cdot (r^2)^{b_0-n+2(q+q')-2i}. \tag{9.120}
\end{aligned}$$

Similarly, for the case of A_2 , we have

$$\begin{aligned}
4A_2 &= (-1)^n \int_0^1 dr^2 (1+r^2)^{a_0} r^{2c_0} \\
&\cdot \left[\frac{e^T \cdot D \cdot k_1}{r} - \frac{e^T \cdot k_1}{1} r \right]^{n-2q} \cdot \left[-\frac{e^T \cdot k_1}{1} r + \frac{e^T \cdot D \cdot k_1}{r} \right]^{n-2q'} \\
&\cdot \left[\frac{e^L \cdot D \cdot k_1}{r^2} + \frac{e^L \cdot k_1}{1} r^2 \right]^q \cdot \left[\frac{e^L \cdot k_1}{1} r^2 + \frac{e^L \cdot D \cdot k_1}{r^2} \right]^{q'} \\
&= (-1)^n (E \sin \phi)^{2n} \left(\frac{2 \cos^2 \frac{\phi}{2}}{M \sin^2 \phi} \right)^{q+q'} \sum_{i=0}^{q+q'} \binom{q+q'}{i} \left(\frac{\sin^2 \frac{\phi}{2}}{\cos^2 \frac{\phi}{2}} \right)^i \\
&\cdot \int_0^1 dr^2 (1+r^2)^{a_0+2n-2(q+q')} (r^2)^{c_0-n+2i}. \tag{9.121}
\end{aligned}$$

The scattering amplitude on RP_2 can therefore be calculated to be

$$\begin{aligned}
A^{RP_2} &= A_1 + A_2 + A_3 + A_4 \\
&= \frac{1}{2} (-1)^n (E \sin \phi)^{2n} \left(\frac{2 \cos^2 \frac{\phi}{2}}{M \sin^2 \phi} \right)^{q+q'} \sum_{i=0}^{q+q'} \binom{q+q'}{i} \left(\frac{\sin^2 \frac{\phi}{2}}{\cos^2 \frac{\phi}{2}} \right)^i \\
&\cdot \int_0^1 dr^2 (1+r^2)^{a_0+2n-2(q+q')} \cdot \left[(r^2)^{b_0-n+2(q+q')-2i} + (r^2)^{c_0-n+2i} \right]. \tag{9.122}
\end{aligned}$$

The integral in Eq.(9.122) can be calculated as following

$$\begin{aligned}
&\int_0^1 dr^2 (1+r^2)^{a_0+2n-2(q+q')} \cdot \left[(r^2)^{b_0-n+2(q+q')-2i} + (r^2)^{c_0-n+2i} \right] \\
&= \left[\frac{2^{1+a_0+2n-2(q+q')}}{1+b_0-n+2(q+q')-2i} \right] \\
&\cdot F(2+a_0+b_0+n-2i, 1, 2+b_0-n+2(q+q')-2i, -1) \\
&+ \left[\frac{2^{1+a_0+2n-2(q+q')}}{1+c_0-n+2i} \right] F(2+a_0+c_0+n-2(q+q')+2i, 1, 2+c_0-n+2i, -1) \tag{9.123}
\end{aligned}$$

where we have used the following identities of the hypergeometric function $F(\alpha, \beta, \gamma, x)$

$$F(\alpha, \beta, \gamma; x) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \int_0^1 dy y^{\beta-1} (1-y)^{\gamma-\beta-1} (1-yx)^{-\alpha}, \tag{9.124}$$

$$F(\alpha, \beta, \gamma, x) = 2^{\gamma-\alpha-\beta} F(\gamma-\alpha, \gamma-\beta, \gamma, x). \tag{9.125}$$

To further reduce the scattering amplitude into beta function, we use the momentum

conservation in Eq.(9.113) and the identity

$$\begin{aligned} & (1 + \alpha) F(-\alpha, 1, 2 + \beta, -1) + (1 + \beta) F(-\beta, 1, 2 + \alpha, -1) \\ &= 2^{1+\alpha+\beta} \frac{\Gamma(\alpha + 2) \Gamma(\beta + 2)}{\Gamma(\alpha + \beta + 2)} \end{aligned} \quad (9.126)$$

to get

$$\begin{aligned} & \left[\frac{2^{1+a_0+2n-2(q+q')}}{1 + b_0 - n + 2(q + q') - 2i} \right] F(-c_0 + n - 2i, 1, 2 + b_0 - n + 2(q + q') - 2i, -1) \\ &+ \left[\frac{2^{1+a_0+2n-2(q+q')}}{1 + c_0 - n + 2i} \right] F(-b_0 + n - 2(q + q') + 2i, 1, 2 + c_0 - n + 2i, -1) \\ &= \frac{\Gamma(1 + c_0 - n + 2i) \Gamma(1 + b_0 - n + 2(q + q') - 2i)}{\Gamma(2 + b_0 + c_0 - 2n + 2(q + q'))} \\ &\sim B(1 + b_0, 1 + c_0) \frac{(1 + c_0)^{-n+2i} (1 + b_0)^{-n+2(q+q')-2i}}{(2 + b_0 + c_0)^{-2n+2(q+q')}} \\ &\sim B(1 + b_0, 1 + c_0) \left(\cos^2 \frac{\phi}{2} \right)^{-n+2i} \left(\sin^2 \frac{\phi}{2} \right)^{-n+2(q+q')-2i}. \end{aligned} \quad (9.127)$$

We finally end up with

$$\begin{aligned} A^{RP_2} &= A_1 + A_2 + A_3 + A_4 \\ &= \frac{1}{2} (-1)^n (E \sin \phi)^{2n} \left(\frac{2 \cos^2 \frac{\phi}{2}}{M \sin^2 \phi} \right)^{q+q'} \sum_{i=0}^{q+q'} \binom{q+q'}{i} \left(\frac{\sin^2 \frac{\phi}{2}}{\cos^2 \frac{\phi}{2}} \right)^i \\ &\cdot B(1 + b_0, 1 + c_0) \left(\cos^2 \frac{\phi}{2} \right)^{-n+2i} \left(\sin^2 \frac{\phi}{2} \right)^{-n+2(q+q')-2i} \\ &= \frac{1}{2} (-1)^n (2E)^{2n} \left(\frac{\sin^2 \frac{\phi}{2}}{2M} \right)^{q+q'} B(1 + b_0, 1 + c_0) \sum_{i=0}^{q+q'} \binom{q+q'}{i} \left(\frac{\cos^2 \frac{\phi}{2}}{\sin^2 \frac{\phi}{2}} \right)^i \\ &= \frac{1}{2} (-1)^n (2E)^{2n} \left(\frac{1}{2M} \right)^{q+q'} B(1 + b_0, 1 + c_0) \\ &\sim \frac{1}{2} (-1)^n (2E)^{2n} \left(\frac{1}{2M} \right)^{q+q'} B\left(-\frac{t}{2}, -\frac{u}{2}\right). \end{aligned} \quad (9.128)$$

From Eq.(9.128) we see that the UV behavior of O-particle scatterings is exponential fall-off and one gets infinite linear relations among string scattering amplitudes of different string states at each fixed mass level. Note that both t and u correspond to the closed string channel poles, while s corresponds to the open string channel poles. It can be seen from Eq.(9.128) that an infinite closed string Regge poles exist in the form factor of O-particle

scatterings. Furthermore, there are no s -channel open string Regge poles as expected since O-planes are not dynamical. This is in contrast to the D-particle scatterings discussed in the last section where both infinite s -channel open string Regge poles and t -channel closed string Regge poles exist in the form factor. We will see that the fundamental characteristics of O-domain-wall scatterings are very different from those of O-particle scatterings as we will now discuss in the next section.

2. Hard strings scattered from O-domain-wall

For this case the kinematic setup is

$$e^P = \frac{1}{M} (-E, k_2 \cos \theta, -k_2 \sin \theta) = \frac{k_2}{M}, \quad (9.129)$$

$$e^L = \frac{1}{M} (-k_2, E \cos \theta, -E \sin \theta), \quad (9.130)$$

$$e^T = (0, \sin \theta, \cos \theta), \quad (9.131)$$

$$k_1 = (E, -k_1 \cos \phi, -k_1 \sin \phi), \quad (9.132)$$

$$k_2 = (-E, k_2 \cos \theta, -k_2 \sin \theta). \quad (9.133)$$

In the high energy limit, the angle of incidence ϕ is identical to the angle of reflection θ and $Diag D_{\mu\nu} = (-1, 1, -1)$. The following kinematic relations can be easily calculated

$$e^T \cdot k_2 = e^L \cdot k_2 = 0, \quad (9.134)$$

$$e^T \cdot k_1 = -2k_1 \sin \phi \cos \phi \sim -E \sin 2\phi, \quad (9.135)$$

$$e^T \cdot D \cdot k_1 = 0, \quad (9.136)$$

$$e^T \cdot D \cdot k_2 = 2k_2 \sin \phi \cos \phi \sim E \sin 2\phi, \quad (9.137)$$

$$e^L \cdot k_1 = \frac{1}{M} [k_2 E - k_1 E (\cos^2 \phi - \sin^2 \phi)] \sim \frac{2E^2}{M} \sin^2 \phi, \quad (9.138)$$

$$e^L \cdot D \cdot k_1 = 0, \quad (9.139)$$

$$e^L \cdot D \cdot k_2 = \frac{1}{M} [-k_2 E + k_2 E (\cos^2 \phi - \sin^2 \phi)] \sim -\frac{2E^2}{M} \sin^2 \phi. \quad (9.140)$$

We define

$$a_0 \equiv k_1 \cdot D \cdot k_1 \sim -2E^2 \sin^2 \phi - 2M_1^2 \cos^2 \phi + M_1^2, \quad (9.141)$$

$$a'_0 \equiv k_2 \cdot D \cdot k_2 = -E^2 - k_2^2 \sim -2E^2, \quad (9.142)$$

$$b_0 \equiv k_1 \cdot k_2 \sim 2E^2 \sin^2 \phi + 2M_1^2 \cos^2 \phi - \frac{1}{2} (M_1^2 + M^2), \quad (9.143)$$

$$c_0 \equiv k_1 \cdot D \cdot k_2 = E^2 - k_1 k_2 \sim \frac{1}{2} (M_1^2 + M^2), \quad (9.144)$$

and the Mandelstam variables can be calculated to be

$$t \equiv -(k_1 + k_2)^2 = M_1^2 + M_2^2 - 2k_1 \cdot k_2 = M_2^2 - 2(1 + b_0), \quad (9.145)$$

$$s \equiv \frac{1}{2} k_1 \cdot D \cdot k_1 = \frac{1}{2} a_0, \quad (9.146)$$

$$u = -2k_1 \cdot D \cdot k_2 = -2c_0. \quad (9.147)$$

The first term of high energy scatterings from O-domain-wall is

$$\begin{aligned} 4A_1 &= (-1)^n \int_0^1 dr^2 (1 + r^2)^{a_0} r^{2b_0} \\ &\cdot \left[\frac{e^T \cdot k_1}{r} - \frac{e^T \cdot D \cdot k_1}{1} r \right]^{n-2q} \cdot \left[-\frac{e^T \cdot D \cdot k_1}{1} r + \frac{e^T \cdot k_1}{r} \right]^{n-2q'} \\ &\cdot \left[\frac{e^L \cdot k_1}{r^2} + \frac{e^L \cdot D \cdot k_1}{1} r^2 \right]^q \cdot \left[\frac{e^L \cdot D \cdot k_1}{1} r^2 + \frac{e^L \cdot k_1}{r^2} \right]^{q'} \\ &\sim (-1)^n (E \sin 2\phi)^{2n} \left(\frac{1}{2M \cos^2 \phi} \right)^{q+q'} \int_0^1 dr^2 (1 + r^2)^{a_0} (r^2)^{b_0-n}. \end{aligned} \quad (9.148)$$

The second term can be similarly calculated to be

$$\begin{aligned} 4A_2 &= (-1)^n \int_0^1 dr^2 (1 + r^2)^{a_0} r^{2c_0} \\ &\cdot \left[\frac{e^T \cdot D \cdot k_1}{r} - \frac{e^T \cdot k_1}{1} r \right]^{n-2q} \cdot \left[-\frac{e^T \cdot k_1}{1} r + \frac{e^T \cdot D \cdot k_1}{r} \right]^{n-2q'} \\ &\cdot \left[\frac{e^L \cdot D \cdot k_1}{r^2} + \frac{e^L \cdot k_1}{1} r^2 \right]^q \cdot \left[\frac{e^L \cdot k_1}{1} r^2 + \frac{e^L \cdot D \cdot k_1}{r^2} \right]^{q'} \\ &\sim (-1)^n (E \sin 2\phi)^{2n-2(q+q')} \left(\frac{2E^2}{M} \sin^2 \phi \right)^{q+q'} \int_0^1 dr^2 (1 + r^2)^{a_0} (r^2)^{c_0+n}. \end{aligned} \quad (9.149)$$

The scattering amplitudes of O-domain-wall on RP_2 can therefore be calculated to be

$$\begin{aligned}
A^{RP_2} &= A_1 + A_2 + A_3 + A_4 \\
&= \frac{1}{2} (-1)^n (E \sin 2\phi)^{2n} \left(\frac{1}{2M \cos^2 \phi} \right)^{q+q'} \\
&\quad \cdot \int_0^1 dr^2 (1+r^2)^{a_0} \left[(r^2)^{b_0-n} + (r^2)^{c_0+n} \right]. \tag{9.150}
\end{aligned}$$

By using the similar technique for the case of O-particle scatterings, the integral above can be calculated to be

$$\begin{aligned}
&\int dr^2 (1+r^2)^{a_0} \left[(r^2)^{b_0-n} + (r^2)^{c_0+n} \right] \\
&= \frac{F(-a_0, 1+b_0-n, 2+b_0-n, -1)}{1+b_0-n} + \frac{F(-a_0, 1+c_0+n, 2+c_0+n, -1)}{1+c_0+n} \\
&= \frac{2^{2+a_0+b_0+c_0}}{(1+b_0-n)(1+c_0+n)} \frac{\Gamma(2+c_0+n) \Gamma(2+b_0-n)}{\Gamma(2+b_0+c_0)} \\
&= \frac{\Gamma(1+c_0+n) \Gamma(1+b_0-n)}{\Gamma(2+b_0+c_0)}. \tag{9.151}
\end{aligned}$$

One thus ends up with

$$\begin{aligned}
A^{RP_2} &= A_1 + A_2 + A_3 + A_4 \\
&= \frac{1}{2} (-1)^n (E \sin 2\phi)^{2n} \left(\frac{1}{2M \cos^2 \phi} \right)^{q+q'} \frac{\Gamma(c_0+n+1) \Gamma(b_0-n+1)}{\Gamma(b_0+c_0+2)}. \tag{9.152}
\end{aligned}$$

Some crucial points of this result are in order. First, since c_0 is a constant in the high energy limit, the UV behavior of the O-domain-wall scatterings is power-law instead of the usual exponential fall-off in other O-plane scatterings.

Second, there exist only *finite* number of closed string poles in the form factor. Note that although we only look at the high energy kinematic regime of the scattering amplitudes, it is easy to see that there exists no infinite closed string Regge poles in the scattering amplitudes for the whole kinematic regime. This is because there is only one kinematic variable for the O-domain-wall scatterings. In fact, the structure of poles in Eq.(9.152) can be calculated to

be

$$\begin{aligned}
& \frac{\Gamma(1+c_0+n)\Gamma(1+b_0-n)}{\Gamma(2+b_0+c_0)} \\
&= \frac{\Gamma(1+M^2)\Gamma(1+b_0-n)}{\Gamma(b_0+n)} \\
&= \Gamma(1+M^2) \frac{(b_0-n)!}{(b_0+n-1)!} \\
&= \Gamma(1+M^2) \prod_{k=1-n}^{n-1} \frac{1}{b_0-k}
\end{aligned} \tag{9.153}$$

where we have used $c_0 \equiv \frac{1}{2}(M_1^2 + M^2)$ in the high energy limit. It is easy to see that the larger the mass M of the external leg is, the more numerous the closed string poles are. We thus confirm that all massive string states do couple to the O-domain-wall as was conjectured previously [51, 52]. This is also consistent with the boundary state descriptions of O-planes.

However, the claim that there is a thickness of order $\sqrt{\alpha'}$ for the O-domain-wall is misleading as the UV behavior of its scatterings is power-law instead of exponential fall-off. This concludes that, in contrast to the usual behavior of high energy, fixed angle string scattering amplitudes, namely soft UV, linear relations and the existence of infinite Regge poles, O-domain-wall scatterings, like the well-known D-instanton scatterings, behave like field theory scatterings.

We summarize the Regge pole structures of closed strings states scattered from various D-branes and O-planes in the table. The s -channel and t -channel scatterings for both D-branes and O-planes are shown in the Fig. 2. For O-plane scatterings, the s -channel open string Regge poles are not allowed since O-planes are not dynamical. For both cases of Domain-wall scatterings, the t -channel closed string Regge poles are not allowed since there is only one kinematic variable instead of two as in the usual cases.

	$p = -1$	$1 \leq p \leq 23$	$p = 24$
D <i>p</i> -branes	X	C+O	O
O <i>p</i> -planes	X	C	X

In this table, "C" and "O" represent infinite Closed string Regge poles and Open string Regge poles respectively. "X" means there are no infinite Regge poles.

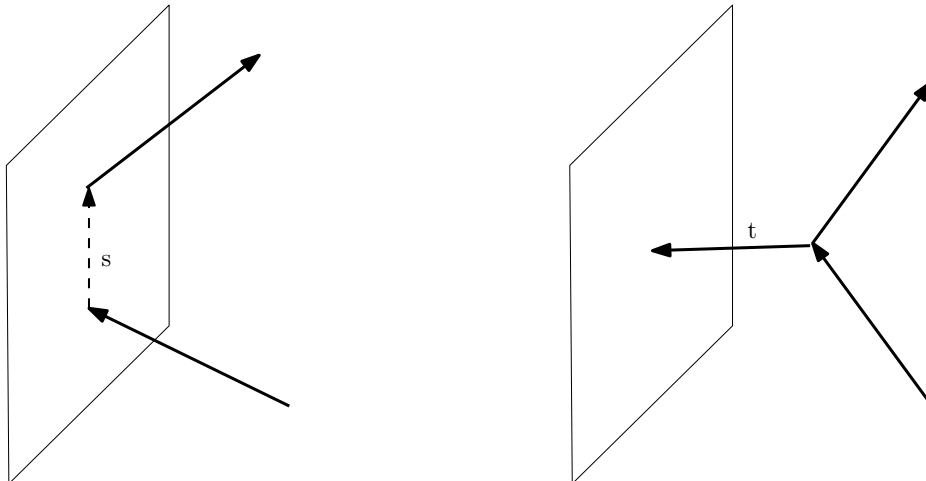


FIG. 2: There are two possible channels for closed strings scattered from D-branes/O-planes. The diagram on the left hand side corresponds to the s-channel scatterings, and the diagram on the right hand side is the t-channel scatterings.

C. Hard closed strings decay to open strings

In this section, we calculate the absorption amplitudes [126] of a closed string state at arbitrary mass level leading to two open string states on the D-brane at high energies. The corresponding simple case of absorption amplitude for massless closed string state was calculated in [140] (The discussion on massless string states scattered from D-brane can be found in [52, 135–139]). The inverse of this process can be used to describe Hawking radiation in the D-brane picture.

As in the case of Domain-wall scattering discussed above, this process contains *one* kinematic variable (energy E) and thus occupies an intermediate position between the conventional three-point and four-point amplitudes. However, in contrast to the power-law behavior of high energy Domain-wall scattering which contains only one kinematic variable (energy E), its form factor behaves as exponential fall-off at high energies.

It is thus of interest to investigate whether the usual linear relations of high energy amplitudes persist for this case or not. As will be shown in this section, after identifying the geometric parameter of the kinematic, one can derive the linear relations (of the kinematic variable) and ratios among the high energy amplitudes corresponding to absorption of different closed string states for each fixed mass level by D-brane. This result is consistent with

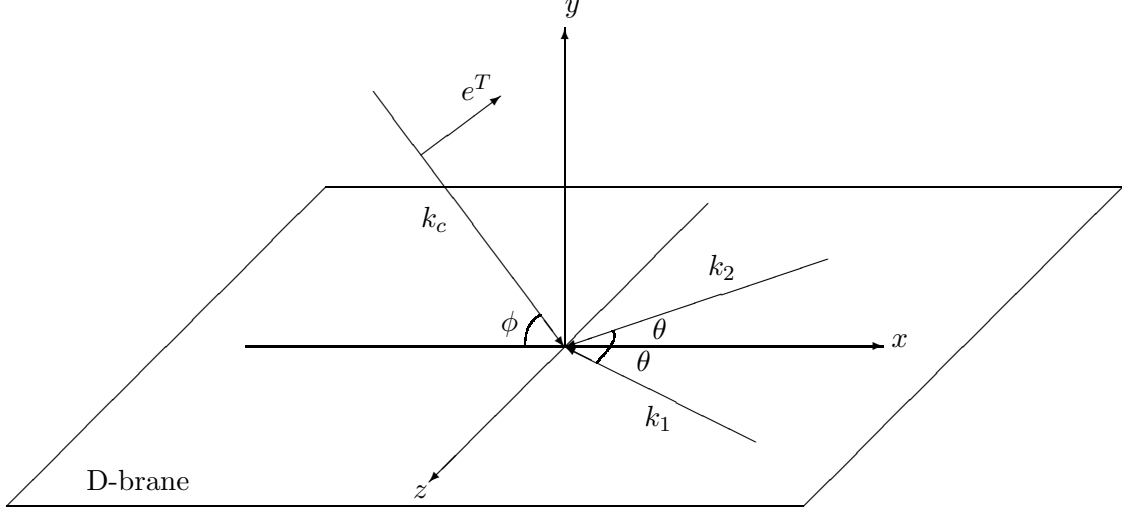


FIG. 3: Kinematic setting up for a closed string decaying to two open strings on a D-brane.

the coexistence [45] of the linear relations and exponential fall-off behavior of high energy string/D-brane amplitudes.

To study the high energy process of Dp brane ($2 \leq p \leq 24$) absorbs (emits) a massive closed string state leading to two open strings on the Dp brane, we set up the kinematic for the massive closed string state to be

$$e^P = \frac{1}{M} (E, k_c \cos \phi, -k_c \sin \phi, 0) = \frac{k_c}{M}, \quad (9.154)$$

$$e^L = \frac{1}{M} (k_c, E \cos \phi, -E \sin \phi, 0), \quad (9.155)$$

$$e^T = (0, \sin \phi, \cos \phi, 0), \quad (9.156)$$

$$k_c = (E, k_c \cos \phi, -k_c \sin \phi, 0). \quad (9.157)$$

For simplicity, we chose the open string excitation to be two tachyons with momenta (see Fig.3)

$$k_1 = \left(-\frac{E}{2}, -\frac{k_{op}}{2} \cos \theta, 0, -\frac{k_{op}}{2} \sin \theta \right), \quad (9.158)$$

$$k_2 = \left(-\frac{E}{2}, -\frac{k_{op}}{2} \cos \theta, 0, +\frac{k_{op}}{2} \sin \theta \right). \quad (9.159)$$

Our final results, however, will remain the same for arbitrary two open string excitation at

high energies. Conservation of momentum on the D-brane implies

$$\underbrace{\frac{1}{2}(k_c + D \cdot k_c)}_{(k_c)_{//}} + k_1 + k_2 = 0 \Rightarrow k_c \cos \phi = k_{op} \cos \theta, \quad (9.160)$$

where $D_{\mu\nu} = \text{diag}\{-1, 1, -1, 1\}$. It is crucial to note that, in the high energy limit, $k_c = k_{op}$ and the scattering angle θ is identical to the incident angle ϕ . One can calculate

$$\begin{aligned} e^T \cdot k_1 &= e^T \cdot k_2 = e^T \cdot D \cdot k_1 = e^T \cdot D \cdot k_2 \\ &= -\frac{k_{op} \cos \theta \sin \phi}{2} = -\frac{k_c \sin \phi \cos \phi}{2}, \end{aligned} \quad (9.161)$$

$$\begin{aligned} e^L \cdot k_1 &= e^L \cdot k_2 = e^L \cdot D \cdot k_1 = e^L \cdot D \cdot k_2 \\ &= \frac{1}{M} \left[\frac{k_c E}{2} - \frac{k_{op} E}{2} \cos \theta \cos \phi \right] = \frac{k_c E}{2M} \sin^2 \phi, \end{aligned} \quad (9.162)$$

$$e^T \cdot D \cdot k_c = 2k_c \sin \phi \cos \phi, \quad (9.163)$$

$$e^L \cdot D \cdot k_c = -\frac{2k_c E}{M} \sin^2 \phi, \quad (9.164)$$

which will be useful for later calculations. We define the kinematic invariants

$$\begin{aligned} t &\equiv -(k_1 + k_2)^2 = M_1^2 + M_2^2 - 2k_1 \cdot k_2 = -2(2 + k_1 \cdot k_2) \\ &= 2k_1 \cdot k_c = 2k_2 \cdot k_c, \end{aligned} \quad (9.165)$$

$$s \equiv 4k_1 \cdot k_2 = 2M_1^2 + 2M_2^2 + 2(k_1 + k_2)^2 = -2(4 + t), \quad (9.166)$$

and calculate the following identities

$$k_1 \cdot k_c + k_2 \cdot D \cdot k_c = k_2 \cdot k_c + k_1 \cdot D \cdot k_c = t, \quad (9.167)$$

$$k_c \cdot D \cdot k_c = M^2 - 2t. \quad (9.168)$$

Note that there is only one kinematic variable as s and t are related in Eq.(9.166) [140]. On the other hand, since the scattering angle θ is fixed by the incident angle ϕ , ϕ and θ are not the dynamical variables in the usual sense.

Following Eq.(9.112), we consider an incoming high energy massive closed state to be [41, 45] $(\alpha_{-1}^T)^{n-m-2q} (\alpha_{-1}^L)^m (\alpha_{-2}^L)^q \otimes (\tilde{\alpha}_{-1}^T)^{n-m'-2q'} (\tilde{\alpha}_{-1}^L)^{m'} (\tilde{\alpha}_{-2}^L)^{q'} |0\rangle$ with $\underline{m = m' = 0}$. The amplitude of the absorption process can be calculated to be

$$\begin{aligned}
A &= \int dx_1 dx_2 d^2 z \cdot (x_1 - x_2)^{k_1 \cdot k_2} (z - \bar{z})^{k_c \cdot D \cdot k_c} (x_1 - z)^{k_1 \cdot k_c} \\
&\cdot (x_1 - \bar{z})^{k_1 \cdot D \cdot k_c} (x_2 - z)^{k_2 \cdot k_c} (x_2 - \bar{z})^{k_2 \cdot D \cdot k_c} \\
&\cdot \exp \left\{ \left\langle \left[ik_1 X(x_1) + ik_2 X(x_2) + ik_c \tilde{X}(\bar{z}) \right] \left[(n-2q) \varepsilon_T^{(1)} \partial X^T + iq \varepsilon_L^{(1)} \partial^2 X^L \right] (z) \right\rangle \right. \\
&+ \left. \left\langle \left[ik_1 X(x_1) + ik_2 X(x_2) + ik_c X(z) \right] \left[(n-2q') \varepsilon_T^{(2)} \bar{\partial} \tilde{X}^T + iq' \varepsilon_L^{(2)} \bar{\partial}^2 \tilde{X}^L \right] (\bar{z}) \right\rangle_{\text{linear terms}} \right\} \\
&= (-1)^{q+q'} \int dx_1 dx_2 d^2 z \cdot (x_1 - x_2)^{k_1 \cdot k_2} (z - \bar{z})^{k_c \cdot D \cdot k_c} (x_1 - z)^{k_1 \cdot k_c} \\
&\cdot (x_1 - \bar{z})^{k_1 \cdot D \cdot k_c} (x_2 - z)^{k_2 \cdot k_c} (x_2 - \bar{z})^{k_2 \cdot D \cdot k_c} \\
&\cdot \left[\frac{ie^T \cdot k_1}{x_1 - z} + \frac{ie^T \cdot k_2}{x_2 - z} + \frac{ie^T \cdot D \cdot k_c}{\bar{z} - z} \right]^{n-2q} \cdot \left[\frac{ie^T \cdot D \cdot k_1}{x_1 - \bar{z}} + \frac{ie^T \cdot D \cdot k_2}{x_2 - \bar{z}} + \frac{ie^T \cdot D \cdot k_c}{z - \bar{z}} \right]^{n-2q'} \\
&\cdot \left[\frac{e^L \cdot k_1}{(x_1 - z)^2} + \frac{e^L \cdot k_2}{(x_2 - z)^2} + \frac{e^L \cdot D \cdot k_c}{(\bar{z} - z)^2} \right]^q \cdot \left[\frac{e^L \cdot D \cdot k_1}{(x_1 - \bar{z})^2} + \frac{e^L \cdot D \cdot k_2}{(x_2 - \bar{z})^2} + \frac{e^L \cdot D \cdot k_c}{(z - \bar{z})^2} \right]^{q'}.
\end{aligned} \tag{9.169}$$

Set $\{x_1, x_2, z\} = \{-x, x, i\}$ to fix the $SL(2, R)$ gauge and use Eq.(9.161-9.164), we have

$$\begin{aligned}
A &= (-1)^{n+M^2/2+t/2} 2^{M^2-2-5t/2} \cdot \int_{-\infty}^{+\infty} dx \cdot x^{-t/2-2} (1-ix)^{t+1} (1+ix)^{t+1} \\
&\cdot \left[\frac{-\frac{k_c \sin \phi \cos \phi}{2}}{1-ix} + \frac{-\frac{k_c \sin \phi \cos \phi}{2}}{1+ix} + \frac{2k_c \sin \phi \cos \phi}{2} \right]^{n-2q} \\
&\cdot \left[\frac{-\frac{k_c \sin \phi \cos \phi}{2}}{1+ix} + \frac{-\frac{k_c \sin \phi \cos \phi}{2}}{1-ix} + \frac{2k_c \sin \phi \cos \phi}{2} \right]^{n-2q'} \\
&\cdot \left[\frac{\frac{k_c E}{2M} \sin^2 \phi}{(1-ix)^2} + \frac{\frac{k_c E}{2M} \sin^2 \phi}{(1+ix)^2} + \frac{-\frac{2k_c E}{M} \sin^2 \phi}{4} \right]^q \cdot \left[\frac{\frac{k_c E}{2M} \sin^2 \phi}{(1+ix)^2} + \frac{\frac{k_c E}{2M} \sin^2 \phi}{(1-ix)^2} + \frac{-\frac{2k_c E}{M} \sin^2 \phi}{4} \right]^{q'} \\
&= (-1)^{n+M^2/2+t/2} 2^{M^2-2-5t/2} \cdot (k_c \sin \phi \cos \phi)^{2n-2(q+q')} \left(-\frac{k_c E \sin^2 \phi}{2M} \right)^{q+q'} \\
&\cdot \int_{-\infty}^{+\infty} dx \cdot x^{-t/2-2} (1+x^2)^{t+1} \left[\frac{x^2}{1+x^2} \right]^{2n-2(q+q')} \left[1 - \frac{2(1-x^2)}{(1+x^2)^2} \right]^{q+q'}.
\end{aligned} \tag{9.170}$$

By using the binomial expansion, we get

$$\begin{aligned}
A &= (-1)^{n+M^2/2+t/2} 2^{M^2-2-5t/2} \cdot (E \sin \phi \cos \phi)^{2n} \left(-\frac{1}{2M \cos^2 \phi} \right)^{q+q'} \\
&\cdot \sum_{i=0}^{q+q'} \sum_{j=0}^i \binom{q+q'}{i} \binom{i}{j} (-2)^i (-1)^j \\
&\int_0^\infty d(x^2) \cdot (x^2)^{-t/4-3/2+2n-2(q+q')+j} (1+x^2)^{t+1-2n+2(q+q')-2i}.
\end{aligned} \tag{9.171}$$

Finally, to reduce the integral to the standard beta function, we do the linear fractional transformation $x^2 = \frac{1-y}{y}$ to get

$$\begin{aligned}
A &= (-1)^{n+M^2/2+t/2} 2^{M^2-2-5t/2} \cdot (E \sin \phi \cos \phi)^{2n} \left(-\frac{1}{2M \cos^2 \phi} \right)^{q+q'} \\
&\cdot \sum_{i=0}^{q+q'} \sum_{j=0}^i \binom{q+q'}{i} \binom{i}{j} (-2)^i (-1)^j \int_0^1 dy \cdot y^{-3t/4-3/2+2i-j} \cdot (1-y)^{-t/4-3/2+2n-2(q+q')+j} \\
&= (-1)^{n+M^2/2+t/2} 2^{M^2-2-5t/2} \cdot (E \sin \phi \cos \phi)^{2n} \left(-\frac{1}{2M \cos^2 \phi} \right)^{q+q'} \\
&\cdot \frac{\Gamma(-\frac{3t}{4}-\frac{1}{2}) \Gamma(-\frac{t}{4}-\frac{1}{2})}{\Gamma(-t-1)} \sum_{i=0}^{q+q'} \sum_{j=0}^i \binom{q+q'}{i} \binom{i}{j} (-2)^i (-1)^j \left(\frac{3}{4} \right)^{2i-j} \left(\frac{1}{4} \right)^{2n-2(q+q')+j} \\
&= (-1)^{n+M^2/2+t/2} 2^{M^2-2-5t/2} \cdot \left(\frac{E \sin \phi \cos \phi}{4} \right)^{2n} \\
&\cdot \left(-\frac{2}{M \cos^2 \phi} \right)^{q+q'} \frac{\Gamma(-\frac{3t}{4}-\frac{1}{2}) \Gamma(-\frac{t}{4}-\frac{1}{2})}{\Gamma(-t-1)}. \tag{9.172}
\end{aligned}$$

In addition to an exponential fall-off factor, the energy E dependence of Eq.(9.172) contains a pre-power factor in the high energy limit. To obtain the linear relations for the amplitudes at each fixed mass level, we rewrite Eq.(9.172) in the following form

$$\frac{\mathcal{T}(n,0,q;n,0,q')}{\mathcal{T}(n,0,0;n,0,0)} = \left(-\frac{2}{M \cos^2 \phi} \right)^{q+q'}. \tag{9.173}$$

One first notes that Eq.(9.173) does not contradict with Eq.(5.60), which predict the ratios $(-\frac{1}{2M})^{q+q'}$. This is because for the absorption process we are considering, there is only one kinematic variable and the usual Ward identity calculations do not apply. To compare Eq.(9.173) with the "ratios" of the Domain-wall scattering [45]

$$\frac{\mathcal{T}(n,0,q;n,0,q')}{\mathcal{T}(n,0,0;n,0,0)} \Big|_{Domain} = \left(\frac{E \sin \phi}{M \sqrt{|M_1^2 - 2M^2 - 1|} \cos^2 \phi} \right)^{q+q'}, \tag{9.174}$$

one sees that, in addition to the incident angle ϕ , there is an energy dependent power factor within the bracket of $q+q'$ in Eq.(9.174) Thus there is no linear relations for the Domain-wall scatterings. On the contrary, Eq.(9.173) gives the linear relations (of the kinematic variable E) and ratios among the high energy amplitudes corresponding to absorption of different closed string states for each fixed mass level n by D-brane.

Note that since the scattering angle θ is fixed by the incident angle ϕ , ϕ is not a dynamical variable in the usual sense. Another way to see this is through the relation of s and t in

Eq.(9.166). We will call such an angle a *geometrical parameter* in contrast to the usual dynamical variable. This kind of geometrical parameter shows up in closed string state scattered from generic Dp -brane (except D-instanton and D-particle) [41, 45]. This is because one has only two dynamical variables for the scatterings, but needs more than two variables to set up the kinematic due to the relative geometry between the D-brane and the scattering plane at high energies.

We emphasize that our result in Eq.(9.173) is consistent with the coexistence [45] of the linear relations and exponential fall-off behavior of high energy string/D-brane amplitudes. That is, linear relations of the amplitudes are responsible for the softer, exponential fall-off high energy string/D-brane scatterings than the power-law field theory scatterings.

X. HARD SCATTERINGS IN COMPACT SPACES

In this chapter, following an old suggestion of Mende [53], we calculate high energy massive scattering amplitudes of bosonic string with some coordinates compactified on the torus [54, 55]. We obtain infinite linear relations among high energy scattering amplitudes of different string states in the Gross kinematic regime (GR). This result is reminiscent of the existence of an infinite number of massive ZNS in the compactified closed and open string spectrums constructed in chapter IV [24, 25].

In addition, we analyze all possible power-law and soft exponential fall-off regimes of high energy compactified bosonic string scatterings by comparing the scatterings with their 26D noncompactified counterparts. In particular, we discover in section X.A the existence of a power-law regime at fixed angle and an exponential fall-off regime at small angle for high energy compactified open string scatterings [55]. These new phenomena never happen in the 26D string scatterings. The linear relations break down as expected in all power-law regimes. The analysis can be extended to the high energy scatterings of the compactified closed string in section X.B, which corrects and extends the results in [54].

A. Open string compactified on torus

1. High energy Scatterings

We consider [55] hard scatterings of 26D open bosonic string with one coordinate compactified on S^1 with radius R . As we will see later, it is straightforward to generalize our calculation to more compactified coordinates. The mode expansion of the compactified coordinate is

$$X^{25}(\sigma, \tau) = x^{25} + K^{25}\tau + i \sum_{k \neq 0} \frac{\alpha_k^{25}}{k} e^{-ik\tau} \cos n\sigma \quad (10.1)$$

where K^{25} is the canonical momentum in the X^{25} direction

$$K^{25} = \frac{2\pi l - \theta_j + \theta_i}{2\pi R}. \quad (10.2)$$

Note that l is the quantized momentum and we have included a nontrivial Wilson line with $U(n)$ Chan-Paton factors, $i, j = 1, 2 \dots n$, which will be important in the later discussion. The mass spectrum of the theory is

$$M^2 = (K^{25})^2 + 2(N - 1) \equiv \left(\frac{2\pi l - \theta_j + \theta_i}{2\pi R} \right)^2 + M^2 \quad (10.3)$$

where we have defined level mass as $M^2 = 2(N - 1)$ and $N = \sum_{k \neq 0} \alpha_{-k}^{25} \alpha_k^{25} + \alpha_{-k}^\mu \alpha_k^\mu$, $\mu = 0, 1, 2 \dots 24$. We are going to consider 4-point correlation function in this chapter. In the center of momentum frame, the kinematic can be set up to be [54]

$$k_1 = \left(+\sqrt{p^2 + M_1^2}, -p, 0, -K_1^{25} \right), \quad (10.4)$$

$$k_2 = \left(+\sqrt{p^2 + M_2^2}, +p, 0, +K_2^{25} \right), \quad (10.5)$$

$$k_3 = \left(-\sqrt{q^2 + M_3^2}, -q \cos \phi, -q \sin \phi, -K_3^{25} \right), \quad (10.6)$$

$$k_4 = \left(-\sqrt{q^2 + M_4^2}, +q \cos \phi, +q \sin \phi, +K_4^{25} \right) \quad (10.7)$$

where p is the incoming momentum, q is the outgoing momentum and ϕ is the center of momentum scattering angle. In the high energy limit, one includes only momenta on the scattering plane, and we have included the fourth component for the compactified direction

as the internal momentum. The conservation of the fourth component of the momenta implies

$$K_1^{25} - K_2^{25} + K_3^{25} - K_4^{25} = 0. \quad (10.8)$$

Note that

$$k_i^2 = K_i^2 - M_i^2 = -M_i^2. \quad (10.9)$$

The center of mass energy E is defined as (for large p, q)

$$E = \frac{1}{2} \left(\sqrt{p^2 + M_1^2} + \sqrt{p^2 + M_2^2} \right) = \frac{1}{2} \left(\sqrt{q^2 + M_3^2} + \sqrt{q^2 + M_4^2} \right). \quad (10.10)$$

We have

$$\begin{aligned} -k_1 \cdot k_2 &= \sqrt{p^2 + M_1^2} \cdot \sqrt{p^2 + M_2^2} + p^2 + K_1^{25} K_2^{25} \\ &= \frac{1}{2} (s + k_1^2 + k_2^2) = \frac{1}{2} s - \frac{1}{2} (M_1^2 + M_2^2), \end{aligned} \quad (10.11)$$

$$\begin{aligned} -k_2 \cdot k_3 &= -\sqrt{p^2 + M_2^2} \cdot \sqrt{q^2 + M_3^2} + pq \cos \phi + K_2^{25} K_3^{25} \\ &= \frac{1}{2} (t + k_2^2 + k_3^2) = \frac{1}{2} t - \frac{1}{2} (M_2^2 + M_3^2), \end{aligned} \quad (10.12)$$

$$\begin{aligned} -k_1 \cdot k_3 &= -\sqrt{p^2 + M_1^2} \cdot \sqrt{q^2 + M_3^2} - pq \cos \phi - K_1^{25} K_3^{25} \\ &= \frac{1}{2} (u + k_1^2 + k_3^2) = \frac{1}{2} u - \frac{1}{2} (M_1^2 + M_3^2) \end{aligned} \quad (10.13)$$

where s, t and u are the Mandelstam variables with

$$s + t + u = \sum_i M_i^2 \sim 2(N - 4). \quad (10.14)$$

Note that the Mandelstam variables defined above are not the usual 25-dimensional Mandelstam variables in the scattering process since we have included the internal momentum K_i^{25} in the definition of k_i . We are now ready to calculate the high energy scattering amplitudes.

In the high energy limit, we define the polarizations on the scattering plane to be

$$e^P = \frac{1}{M_2} \left(\sqrt{p^2 + M_2^2}, p, 0, 0 \right), \quad (10.15)$$

$$e^L = \frac{1}{M_2} \left(p, \sqrt{p^2 + M_2^2}, 0, 0 \right), \quad (10.16)$$

$$e^T = (0, 0, 1, 0) \quad (10.17)$$

where the fourth component refers to the compactified direction. It is easy to calculate the following relations

$$e^P \cdot k_1 = -\frac{1}{M_2} \left(\sqrt{p^2 + M_1^2} \sqrt{p^2 + M_2^2} + p^2 \right), \quad (10.18)$$

$$e^P \cdot k_3 = \frac{1}{M_2} \left(\sqrt{q^2 + M_3^2} \sqrt{p^2 + M_2^2} - pq \cos \phi \right), \quad (10.19)$$

$$e^L \cdot k_1 = -\frac{p}{M_2} \left(\sqrt{p^2 + M_1^2} + \sqrt{p^2 + M_2^2} \right), \quad (10.20)$$

$$e^L \cdot k_3 = \frac{1}{M_2} \left(p \sqrt{q^2 + M_3^2} - q \sqrt{p^2 + M_2^2} \cos \phi \right), \quad (10.21)$$

$$e^T \cdot k_1 = 0, \quad e^T \cdot k_3 = -q \sin \phi. \quad (10.22)$$

In this chapter, we will consider the case of a tensor state [54]

$$(\alpha_{-1}^T)^{N-2r} (\alpha_{-2}^L)^r |k_2, l_2, i, j\rangle \quad (10.23)$$

at a general mass level $M_2^2 = 2(N-1)$ scattered with three "tachyon" states (with $M_1^2 = M_3^2 = M_4^2 = -2$). In general, we could have considered the more general high energy state

$$(\alpha_{-1}^T)^{N-2r-2m-\sum_n n s_n} (\alpha_{-1}^L)^{2m} (\alpha_{-2}^L)^r \prod_n (\alpha_{-n}^{25})^{s_n} |k_2, l_2, i, j\rangle. \quad (10.24)$$

However, for our purpose here and for simplicity, we will not consider the general vertex in this chapter. The $s-t$ channel of the high energy scattering amplitude can be calculated to be (We will ignore the trace factor due to Chan-Paton in the scattering amplitude calculation . This does not affect our final results in this chapter)

$$\begin{aligned} A &= \int d^4x \left\langle e^{ik_1 X}(x_1) (\partial X^T)^{N-2r} (i\partial^2 X^L)^r e^{ik_2 X}(x_2) e^{ik_3 X}(x_3) e^{ik_4 X}(x_4) \right\rangle \\ &= \int d^4x \cdot \prod_{i < j} (x_i - x_j)^{k_i \cdot k_j} \\ &\quad \cdot \left[\frac{ie^T \cdot k_1}{x_1 - x_2} + \frac{ie^T \cdot k_3}{x_3 - x_2} + \frac{ie^T \cdot k_4}{x_4 - x_2} \right]^{N-2r} \cdot \left[\frac{e^L \cdot k_1}{(x_1 - x_2)^2} + \frac{e^L \cdot k_3}{(x_3 - x_2)^2} + \frac{e^L \cdot k_4}{(x_4 - x_2)^2} \right]^r. \end{aligned} \quad (10.25)$$

After fixing the $SL(2, R)$ gauge and using the kinematic relations derived previously, we

have

$$\begin{aligned}
A &= i^N (-1)^{k_1 \cdot k_2 + k_1 \cdot k_3 + k_2 \cdot k_3} (q \sin \phi)^{N-2r} \left(\frac{1}{M_2} \right)^r \cdot \int_0^1 dx \cdot x^{k_1 \cdot k_2} (1-x)^{k_2 \cdot k_3} \cdot \left[\frac{1}{1-x} \right]^{N-2r} \\
&\cdot \left[\frac{\left(\sqrt{p^2 + M_1^2} + \sqrt{p^2 + M_2^2} \right)}{x^2} - \frac{\left(p\sqrt{q^2 + M_3^2} - q\sqrt{p^2 + M_2^2} \cos \phi \right)}{(1-x)^2} \right]^r \\
&= (-1)^{k_1 \cdot k_2 + k_1 \cdot k_3 + k_2 \cdot k_3} (iq \sin \phi)^N \left(-\frac{\left(p\sqrt{q^2 + M_3^2} - q\sqrt{p^2 + M_2^2} \cos \phi \right)}{M_2 q^2 \sin^2 \phi} \right)^r \\
&\cdot \sum_{i=0}^r \binom{r}{i} \left[-\frac{\left(\sqrt{p^2 + M_1^2} + \sqrt{p^2 + M_2^2} \right)}{\left(p\sqrt{q^2 + M_3^2} - q\sqrt{p^2 + M_2^2} \cos \phi \right)} \right]^i \cdot \int_0^1 dx \cdot x^{k_1 \cdot k_2 - 2i} (1-x)^{k_2 \cdot k_3 - N + 2i} \\
&= (-iq \sin \phi)^N \left(-\frac{\left(p\sqrt{q^2 + M_3^2} - q\sqrt{p^2 + M_2^2} \cos \phi \right)}{M_2 q^2 \sin^2 \phi} \right)^r \\
&\cdot \sum_{i=0}^r \binom{r}{i} \left[-\frac{\left(\sqrt{p^2 + M_1^2} \sqrt{p^2 + M_2^2} + p^2 \right)}{\left(p\sqrt{q^2 + M_3^2} - q\sqrt{p^2 + M_2^2} \cos \phi \right)} \right]^i \cdot B \left(-\frac{1}{2}s + N - 2i - 1, -\frac{1}{2}t + 2i - 1 \right)
\end{aligned} \tag{10.26}$$

where $B(u, v)$ is the Euler beta function. We can do the high energy approximation of the

gamma function $\Gamma(x)$ then do the summation, and end up with

$$\begin{aligned}
A &= (-iq \sin \phi)^N \left(-\frac{(p\sqrt{q^2 + M_3^2} - q\sqrt{p^2 + M_2^2} \cos \phi)}{M_2 q^2 \sin^2 \phi} \right)^r \\
&\cdot \sum_{i=0}^r \binom{r}{i} \left[-\frac{(\sqrt{p^2 + M_1^2} + \sqrt{p^2 + M_2^2})}{(p\sqrt{q^2 + M_3^2} - q\sqrt{p^2 + M_2^2} \cos \phi)} \right]^i \cdot \frac{\Gamma(-1 - \frac{1}{2}s + N - 2i) \Gamma(-1 - \frac{1}{2}t + 2i)}{\Gamma(2 + \frac{1}{2}u)} \\
&\simeq (-iq \sin \phi)^N \left(-\frac{(p\sqrt{q^2 + M_3^2} - q\sqrt{p^2 + M_2^2} \cos \phi)}{M_2 q^2 \sin^2 \phi} \right)^r \\
&\cdot \sum_{i=0}^r \binom{r}{i} \left[-\frac{(\sqrt{p^2 + M_1^2} + \sqrt{p^2 + M_2^2})}{(p\sqrt{q^2 + M_3^2} - q\sqrt{p^2 + M_2^2} \cos \phi)} \right]^i \\
&\cdot B\left(-1 - \frac{1}{2}s, -1 - \frac{1}{2}t\right) \left(-1 - \frac{1}{2}s\right)^{N-2i} \left(-1 - \frac{1}{2}t\right)^{2i} \left(2 + \frac{1}{2}u\right)^{-N} \\
&= \left(-iq \frac{\sin \frac{\phi}{2}}{\cos \frac{\phi}{2}}\right)^N \left(-\frac{1}{M_2}\right)^r \cdot B\left(-1 - \frac{1}{2}s, -1 - \frac{1}{2}t\right) \\
&\cdot \left[\frac{(p\sqrt{q^2 + M_3^2} - q\sqrt{p^2 + M_2^2} \cos \phi)}{q^2 \sin^2 \phi} - \frac{(\sqrt{p^2 + M_1^2} + \sqrt{p^2 + M_2^2})}{q^2 \sin^2 \phi} \left(\frac{t}{s}\right)^2 \right]^r. \quad (10.27)
\end{aligned}$$

2. Classification of Compactified String Scatterings

It is well known that there are two kinematic regimes for the high energy string scatterings in 26D open bosonic string theory. The UV behavior of the finite and fixed angle scatterings in the GR is soft exponential fall-off. Moreover, there exist infinite linear relations among scatterings of different string states in this regime [26–31, 33, 35, 41, 44, 126]. On the other hand, the UV behavior of the small angle scatterings in the Regge regime is hard power-law. The linear relations break down in the Regge regime. As we will see soon, the UV structure of the compactified open string scatterings is more richer.

In this section, we systematically analyze all possible power-law regimes of high energy compactified open string scatterings by comparing the scatterings with their noncompactified counterparts. In particular, we show that all hard power-law regimes of high energy compactified open string scatterings can be traced back to the Regge regime of the 26D high energy string scatterings. The linear relations break down as expected in all power-law regimes. The analysis can be extended to the high energy scatterings of the compactified

closed string in section X.B, which corrects and extends the results in [54].

a. Gross Regime - Linear Relations In the Gross regime, $p^2 \simeq q^2 \gg K_i^2$ and $p^2 \simeq q^2 \gg N$, Eq.(10.27) reduces to

$$A \simeq \left(-iE \frac{\sin \frac{\phi}{2}}{\cos \frac{\phi}{2}} \right)^N \left(-\frac{1}{2M_2} \right)^r \cdot B \left(-1 - \frac{1}{2}s, -1 - \frac{1}{2}t \right). \quad (10.28)$$

For each fixed mass level N , we have the linear relation for the scattering amplitudes

$$\frac{\mathcal{T}^{(n,r)}}{\mathcal{T}^{(n,0)}} = \left(-\frac{1}{2M_2} \right)^r \quad (10.29)$$

with coefficients consistent with our previous results [26–31, 33, 35, 41, 44, 126]. Note that in Eq.(10.28) there is an exponential fall-off factor in the high energy expansion of the beta function. The infinite linear relation in Eq.(10.29) "soften" the high energy behavior of string scatterings in the GR.

b. Classification of compactified open string Since our definitions of the Mandelstam variables s, t and u in Eq.(10.11) to Eq.(10.13) include the compactified coordinates, we can analyze the UV structure of the compactified string scatterings by comparing the scatterings with their simpler 26D counterparts. We introduce the space part of the momentum vectors

$$\mathbf{k}_1 = (-p, 0, -K_1^{25}), \quad (10.30)$$

$$\mathbf{k}_2 = (+p, 0, +K_2^{25}), \quad (10.31)$$

$$\mathbf{k}_3 = (-q \cos \phi, -q \sin \phi, -K_3^{25}), \quad (10.32)$$

$$\mathbf{k}_4 = (+q \cos \phi, +q \sin \phi, +K_4^{25}), \quad (10.33)$$

and define the "26D scattering angle" $\tilde{\phi}$ as following

$$\mathbf{k}_1 \cdot \mathbf{k}_3 = |\mathbf{k}_1| |\mathbf{k}_3| \cos \tilde{\phi}. \quad (10.34)$$

It is then easy to see that the UV behavior of the compactified string scatterings is power-law if and only if $\tilde{\phi}$ is small. This criterion can be used to classify all possible power-law and exponential fall-off kinematic regimes of high energy compactified open string scatterings.

c. Compactified 25D scatterings We first consider the high energy scatterings with only one coordinate compactified.

I. For the case of $\phi = \text{finite}$, the only choice to achieve UV power-law behavior is to require (we choose $K_1^{25} \simeq K_2^{25} \simeq K_3^{25} \simeq K_4^{25}$ and $p \simeq q$ in the following discussion)

$$(K_i^{25})^2 \gg p^2 \simeq q^2 \gg N. \quad (10.35)$$

By the criterion of Eq.(10.34), this is a power-law regime. To explicitly show that this choice of kinematic regime does lead to UV power-law behavior, we will show that it implies

$$s = \text{constant} \quad (10.36)$$

in the open string scattering amplitudes, which in turn gives the desire power-law behavior of high energy compactified open string scattering in Eq.(10.27). On the other hand, it can be shown that the linear relations break down as expected in this regime. For the choice of kinematic regime in Eq.(10.35), Eq.(10.11) and Eq.(10.36) imply

$$\lim_{p \rightarrow \infty} \frac{\sqrt{p^2 + M_1^2} \cdot \sqrt{p^2 + M_2^2} + p^2}{K_1^{25} K_2^{25}} = \lim_{p \rightarrow \infty} \frac{\sqrt{p^2 + M_1^2} \cdot \sqrt{p^2 + M_2^2} + p^2}{\left(\frac{2\pi l_1 - \theta_{j,1} + \theta_{i,1}}{2\pi R}\right) \left(\frac{2\pi l_2 - \theta_{j,2} + \theta_{i,2}}{2\pi R}\right)} = -1. \quad (10.37)$$

For finite momenta l_1 and l_2 , Eq.(10.37) can be achieved by scattering of string states with "super-highly" winding nontrivial Wilson lines

$$(\theta_{i,1} - \theta_{j,1}) \rightarrow \infty, \quad (\theta_{i,2} - \theta_{j,2}) \rightarrow -\infty. \quad (10.38)$$

A careful analysis for this choice gives

$$(\lambda_1 + \lambda_2)^2 = 0 \quad (10.39)$$

where signs of $\lambda_1 = \frac{p}{K_1^{25}}$ and $\lambda_2 = -\frac{p}{K_2^{25}}$ are chosen to be the same. It can be seen now that the kinematic regime in Eq.(10.35) does solve Eq.(10.39).

We now consider the second possible regime for the case of $\phi = \text{finite}$, namely

$$(K_i^{25})^2 \simeq p^2 \simeq q^2 \gg N. \quad (10.40)$$

By the criterion of Eq.(10.34), this is an exponential fall-off regime. To explicitly show that this choice of kinematic regime does lead to UV exponential fall-off behavior, we see that, for this regime, it is impossible to achieve Eq.(10.36) since Eq.(10.39) has no nontrivial solution. Note that there are no linear relations in this regime. Although $\tilde{\phi} = \text{finite}$ in this regime, it is different from the GR in the 26D scatterings since K_i^{25} is as big as the scattering energy

p . In conclusion, we have discovered a $\phi = \text{finite}$ regime with UV power-law behavior for the high energy compactified open string scatterings. This new phenomenon never happens in the 26D string scatterings. The linear relations break down as expected in this regime.

II. For the case of small angle $\phi \simeq 0$ scattering, we consider the first power-law regime

$$qK_1^{25} = -pK_3^{25} \text{ and } (K_i^{25})^2 \simeq p^2 \simeq q^2 \gg N. \quad (10.41)$$

By the criterion of Eq.(10.34), this is a power-law regime. To explicitly show that this choice of kinematic regime does lead to UV power-law behavior, we will show that it implies

$$u = \text{constant} \quad (10.42)$$

in the open string scattering amplitudes, which in turn gives the desire power-law behavior of high energy compactified open string scattering in Eq.(10.27). For this choice of kinematic regime, Eq.(10.13) and Eq.(10.41) imply

$$\lim_{p \rightarrow \infty} \frac{\sqrt{p^2 + M_1^2} \cdot \sqrt{q^2 + M_3^2} + pq}{K_1^{25} K_3^{25}} = -1. \quad (10.43)$$

By choosing different sign for K_1^{25} and K_3^{25} , Eq.(10.43) can be solved for any real number $\lambda \equiv \frac{p}{K_1^{25}} = -\frac{q}{K_3^{25}}$.

The second choice for the power-law regime is the same as Eq.(10.35) in the $\phi = \text{finite}$ regime. The proof to show that it is indeed a power-law regime is similar to the proof in section I.

The last choice for the power-law regime is

$$(K_i^{25})^2 \ll p^2 \simeq q^2 \gg N. \quad (10.44)$$

It is easy to show that this is indeed a power-law regime.

The last kinematic regime for the case of small angle $\phi \simeq 0$ scattering is

$$qK_1^{25} \neq -pK_3^{25} \text{ and } (K_i^{25})^2 \simeq p^2 \simeq q^2 \gg N. \quad (10.45)$$

By the criterion of Eq.(10.34), this is an exponential fall-off regime. We give one example here. Let's choose $\lambda \equiv \frac{p}{K_1^{25}} \neq -\frac{q}{K_3^{25}} = 2\lambda$. By choosing different sign for K_1^{25} and K_3^{25} , Eq.(10.43) reduces to

$$\lambda^2 = 0, \quad (10.46)$$

which has no nontrivial solution for λ , and one can not achieve the power-law condition Eq.(10.42). So this is an exponential fall-off regime. In conclusion, we have discovered a $\phi \simeq 0$ regime with UV exponential fall-off behavior for the high energy compactified open string scatterings. This new phenomenon never happens in the 26D string scatterings. This completes the classification of all kinematic regimes for compactified 25D scatterings.

Compactified 24D (or less) scatterings For this case, we need to introduce another parameter to classify the UV behavior of high energy scatterings, namely the angle δ between \vec{K}_1 and \vec{K}_2 , $\vec{K}_1 \cdot \vec{K}_2 = |K_1| |K_2| \cos \delta$. Similar results can be easily derived through the same method used in the compactified 25D scatterings. The classification is independent of the details of the moduli space of the compact spaces. We summarize the results in the following table:

ϕ	$\tilde{\phi}$	UV Behavior	Examples of the Kinematic Regimes	Linear Relations
finite	finite	Exponential fall-off	$\vec{K}_i^2 \ll p^2 \simeq q^2 \gg N$	Yes
			$\vec{K}_i^2 \simeq p^2 \simeq q^2 \gg N$	No
			$\vec{K}_i^2 \gg p^2 \simeq q^2 \gg N$ and $\cos \delta \neq 0$	
	$\tilde{\phi} \simeq 0$	Power-law	$\vec{K}_i^2 \gg p^2 \simeq q^2 \gg N$ and $\cos \delta = 0$	
$\phi \simeq 0$	$\tilde{\phi} \simeq 0$	Power-law	$\vec{K}_i^2 \ll p^2 \simeq q^2 \gg N$	No.
			$\vec{K}_i^2 \simeq p^2 \simeq q^2 \gg N$ and $q\vec{K}_1 = -p\vec{K}_3$	
			$\vec{K}_i^2 \gg p^2 \simeq q^2 \gg N$ and $\cos \delta = 0$	
	finite	Exponential fall-off	$\vec{K}_i^2 \simeq p^2 \simeq q^2 \gg N$ and $q\vec{K}_1 \neq -p\vec{K}_3$	
			$\vec{K}_i^2 \gg p^2 \simeq q^2 \gg N$ and $\cos \delta \neq 0$	

B. Closed string compactified on torus

In this section, we consider hard scatterings of 26D closed bosonic string [54] with one coordinate compactified on S^1 with radius R . As we will see later, it is straightforward to generalize our calculation to more compactified coordinates.

1. Winding string and kinematic setup

The closed string boundary condition for the compactified coordinate is (we use the notation in [9])

$$X^{25}(\sigma + 2\pi, \tau) = X^{25}(\sigma, \tau) + 2\pi R n, \quad (10.47)$$

where n is the winding number. The momentum in the X^{25} direction is then quantized to be

$$K = \frac{m}{R}, \quad (10.48)$$

where m is an integer. The mode expansion of the compactified coordinate is

$$X^{25}(\sigma, \tau) = X_R^{25}(\sigma - \tau) + X_L^{25}(\sigma + \tau), \quad (10.49)$$

where

$$X_R^{25}(\sigma - \tau) = \frac{1}{2}x + K_R(\sigma - \tau) + i \sum_{k=0} \frac{1}{k} \alpha_k^{25} e^{-2ik(\sigma - \tau)}, \quad (10.50)$$

$$X_L^{25}(\sigma + \tau) = \frac{1}{2}x + K_L(\sigma + \tau) + i \sum_{k=0} \frac{1}{k} \tilde{\alpha}_k^{25} e^{-2ik(\sigma + \tau)}. \quad (10.51)$$

The left and right momenta are defined to be

$$K_{L,R} = K \pm L = \frac{m}{R} \pm \frac{1}{2}nR \Rightarrow K = \frac{1}{2}(K_L + K_R), \quad (10.52)$$

and the mass spectrum can be calculated to be

$$\left\{ \begin{array}{l} M^2 = \left(\frac{m^2}{R^2} + \frac{1}{4}n^2R^2 \right) + N_R + N_L - 2 \equiv K_L^2 + M_L^2 \equiv K_R^2 + M_R^2, \\ N_R - N_L = mn \end{array} \right., \quad (10.53)$$

where N_R and N_L are the number operators for the right and left movers, which include the counting of the compactified coordinate. We have also introduced the left and the right level masses as

$$M_{L,R}^2 \equiv 2(N_{L,R} - 1). \quad (10.54)$$

Note that for the compactified closed string N_R and N_L are correlated through the winding modes.

In the center of momentum frame, the kinematic can be set up to be

$$k_{1L,R} = \left(+\sqrt{p^2 + M_1^2}, -p, 0, -K_{1L,R} \right), \quad (10.55)$$

$$k_{2L,R} = \left(+\sqrt{p^2 + M_2^2}, +p, 0, +K_{2L,R} \right), \quad (10.56)$$

$$k_{3L,R} = \left(-\sqrt{q^2 + M_3^2}, -q \cos \phi, -q \sin \phi, -K_{3L,R} \right), \quad (10.57)$$

$$k_{4L,R} = \left(-\sqrt{q^2 + M_4^2}, +q \cos \phi, +q \sin \phi, +K_{4L,R} \right) \quad (10.58)$$

where $p \equiv |\tilde{p}|$ and $q \equiv |\tilde{q}|$ and

$$k_i \equiv \frac{1}{2} (k_{iR} + k_{iL}), \quad (10.59)$$

$$k_i^2 = K_i^2 - M_i^2, \quad (10.60)$$

$$k_{iL,R}^2 = K_{iL,R}^2 - M_i^2 \equiv -M_{iL,R}^2. \quad (10.61)$$

With this setup, the center of mass energy E is

$$E = \frac{1}{2} \left(\sqrt{p^2 + M_1^2} + \sqrt{p^2 + M_2^2} \right) = \frac{1}{2} \left(\sqrt{q^2 + M_3^2} + \sqrt{q^2 + M_4^2} \right). \quad (10.62)$$

The conservation of momentum on the compactified direction gives

$$m_1 - m_2 + m_3 - m_4 = 0, \quad (10.63)$$

and T-duality symmetry implies conservation of winding number

$$n_1 - n_2 + n_3 - n_4 = 0. \quad (10.64)$$

One can easily calculate the following kinematic relations

$$-k_{1L,R} \cdot k_{2L,R} = \sqrt{p^2 + M_1^2} \cdot \sqrt{p^2 + M_2^2} + p^2 + \vec{K}_{1L,R} \cdot \vec{K}_{2L,R} \quad (10.65)$$

$$= \frac{1}{2} (s_{L,R} + k_{1L,R}^2 + k_{2L,R}^2) = \frac{1}{2} s_{L,R} - \frac{1}{2} (M_{1L,R}^2 + M_{2L,R}^2), \quad (10.66)$$

$$-k_{2L,R} \cdot k_{3L,R} = -\sqrt{p^2 + M_2^2} \cdot \sqrt{q^2 + M_3^2} + pq \cos \phi + \vec{K}_{2L,R} \cdot \vec{K}_{3L,R} \quad (10.67)$$

$$= \frac{1}{2} (t_{L,R} + k_{2L,R}^2 + k_{3L,R}^2) = \frac{1}{2} t_{L,R} - \frac{1}{2} (M_{2L,R}^2 + M_{3L,R}^2), \quad (10.68)$$

$$-k_{1L,R} \cdot k_{3L,R} = -\sqrt{p^2 + M_1^2} \cdot \sqrt{q^2 + M_3^2} - pq \cos \phi - \vec{K}_{1L,R} \cdot \vec{K}_{3L,R} \quad (10.69)$$

$$= \frac{1}{2} (u_{L,R} + k_{1L,R}^2 + k_{3L,R}^2) = \frac{1}{2} u_{L,R} - \frac{1}{2} (M_{1L,R}^2 + M_{3L,R}^2) \quad (10.70)$$

where the left and the right Mandelstam variables are defined to be

$$s_{L,R} \equiv -(k_{1L,R} + k_{2L,R})^2, \quad (10.71)$$

$$t_{L,R} \equiv -(k_{2L,R} + k_{3L,R})^2, \quad (10.72)$$

$$u_{L,R} \equiv -(k_{1L,R} + k_{3L,R})^2, \quad (10.73)$$

with

$$s_{L,R} + t_{L,R} + u_{L,R} = \sum_i M_{iL,R}^2. \quad (10.74)$$

2. Four-tachyon scatterings with $N_R = N_L = 0$

We are now ready to calculate the string scattering amplitudes. Let's first calculate the case with $N_R + N_L = 0$ (or $N_R = N_L = 0$),

$$\begin{aligned} & A_{\text{closed}}^{(N_R+N_L=0)}(s, t, u) \\ &= \int d^2z \exp \{k_{1L} \cdot k_{2L} \ln z + k_{1R} \cdot k_{2R} \ln \bar{z} + k_{2L} \cdot k_{3L} \ln (1-z) + k_{2R} \cdot k_{3R} \ln (1-\bar{z})\} \\ &= \int d^2z \exp \{2k_{1R} \cdot k_{2R} \ln |z| + 2k_{2R} \cdot k_{3R} \ln |1-z| \\ &\quad + (k_{1L} \cdot k_{2L} - k_{1R} \cdot k_{2R}) \ln z + (k_{2L} \cdot k_{3L} - k_{2R} \cdot k_{3R}) \ln (1-z)\} \end{aligned} \quad (10.75)$$

where we have used $\alpha' = 2$ for closed string propagators

$$\langle X(z) X(z') \rangle = -\frac{\alpha'}{2} \ln(z - z'), \quad (10.76)$$

$$\langle \tilde{X}(\bar{z}) \tilde{X}(\bar{z}') \rangle = -\frac{\alpha'}{2} \ln(\bar{z} - \bar{z}'). \quad (10.77)$$

Note that for this simple case, Eq.(10.53) implies either $m = 0$ or $n = 0$. However, we will keep track of the general values of (m, n) here for the reference of future calculations. By using the formula [96]

$$\begin{aligned} I &= \int \frac{d^2z}{\pi} |z|^\alpha |1-z|^\beta z^n (1-z)^m \\ &= \frac{\Gamma(-1 - \frac{1}{2}\alpha - \frac{1}{2}\beta) \Gamma(1+n + \frac{1}{2}\alpha) \Gamma(1+m + \frac{1}{2}\beta)}{\Gamma(-\frac{1}{2}\alpha) \Gamma(-\frac{1}{2}\beta) \Gamma(2+n+m + \frac{1}{2}\alpha + \frac{1}{2}\beta)}, \end{aligned} \quad (10.78)$$

we obtain

$$\begin{aligned}
& A_{\text{closed}}^{(N_R+N_L=0)}(s, t, u) \\
&= \pi \frac{\Gamma(-1 - k_{1R} \cdot k_{2R} - k_{2R} \cdot k_{3R}) \Gamma(1 + k_{1L} \cdot k_{2L}) \Gamma(1 + k_{2L} \cdot k_{3L})}{\Gamma(-k_{1R} \cdot k_{2R}) \Gamma(-k_{2R} \cdot k_{3R}) \Gamma(2 + k_{1L} \cdot k_{2L} + k_{2L} \cdot k_{3L})} \\
&= \frac{\sin(-\pi k_{1R} \cdot k_{2R}) \sin(-\pi k_{2R} \cdot k_{3R})}{\sin(-\pi - \pi k_{1R} \cdot k_{2R} - \pi k_{2R} \cdot k_{3R})} \\
&\quad \cdot \frac{\Gamma(1 + k_{1R} \cdot k_{2R}) \Gamma(1 + k_{2R} \cdot k_{3R})}{\Gamma(2 + k_{1R} \cdot k_{2R} + k_{2R} \cdot k_{3R})} \frac{\Gamma(1 + k_{1L} \cdot k_{2L}) \Gamma(1 + k_{2L} \cdot k_{3L})}{\Gamma(2 + k_{1L} \cdot k_{2L} + k_{2L} \cdot k_{3L})} \\
&\simeq \frac{\sin(\pi s_L/2) \sin(\pi t_R/2)}{\sin(\pi u_L/2)} \frac{\Gamma(-1 - \frac{t_R}{2}) \Gamma(-1 - \frac{u_R}{2})}{\Gamma(2 + \frac{s_R}{2})} \frac{\Gamma(-1 - \frac{t_L}{2}) \Gamma(-1 - \frac{u_L}{2})}{\Gamma(2 + \frac{s_L}{2})}, \quad (10.79)
\end{aligned}$$

where we have used $M_{iL,R}^2 = -2$ for $i = 1, 2, 3, 4$. In the above calculation, we have used the following well known formula for gamma function

$$\Gamma(x) = \frac{\pi}{\sin(\pi x) \Gamma(1-x)}. \quad (10.80)$$

3. High energy massive scatterings for general $N_R + N_L$

We now proceed to calculate the high energy scattering amplitudes for general higher mass levels with fixed $N_R + N_L$. With one compactified coordinate, the mass spectrum of the second vertex of the amplitude is

$$M_2^2 = \left(\frac{m_2^2}{R^2} + \frac{1}{4} n_2^2 R^2 \right) + N_R + N_L - 2. \quad (10.81)$$

We now have more mass parameters to define the "high energy limit". So let's first clear and redefine the concept of "high energy limit" in our following calculations. We are going to use three quantities E^2 , M_2^2 and $N_R + N_L$ to define different regimes of "high energy limit". See FIG. 4. The high energy regime defined by $E^2 \simeq M_2^2 \gg N_R + N_L$ will be called Mende regime (MR). The high energy regime defined by $E^2 \gg M_2^2$, $E^2 \gg N_R + N_L$ will be called Gross region (GR). In the high energy limit, the polarizations on the scattering plane for the second vertex operator are defined to be

$$e^p = \frac{1}{M_2} \left(\sqrt{p^2 + M_2^2}, p, 0, 0 \right), \quad (10.82)$$

$$e^L = \frac{1}{M_2} \left(p, \sqrt{p^2 + M_2^2}, 0, 0 \right), \quad (10.83)$$

$$e^T = (0, 0, 1, 0) \quad (10.84)$$

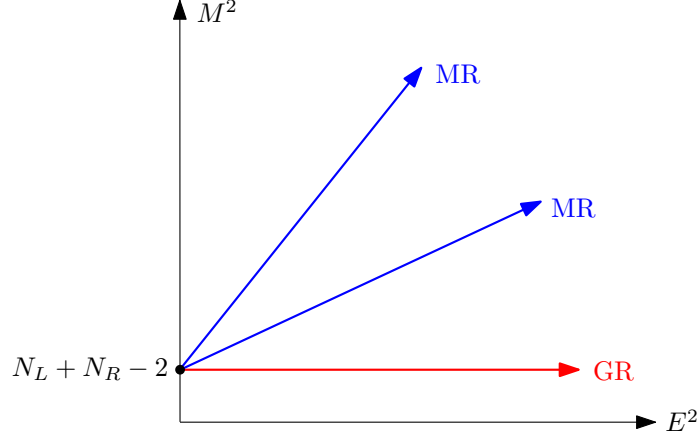


FIG. 4: Different regimes of "high energy limit". The high energy regime defined by $E^2 \simeq M_2^2 \gg N_R + N_L$ will be called Mende regime (MR). The high energy regime defined by $E^2 \gg M_2^2 \simeq N_R + N_L$ will be called Gross region (GR).

where the fourth component refers to the compactified direction. One can calculate the following formulas in the high energy limit

$$\begin{aligned} e^p \cdot k_{1L} = e^p \cdot k_{1R} &= -\frac{1}{M_2} \left(\sqrt{p^2 + M_1^2} \sqrt{p^2 + M_2^2} + p^2 \right) \\ &= -\frac{p^2}{M_2} \left(2 + \frac{M_1^2}{2p^2} + \frac{M_2^2}{2p^2} \right) + \mathcal{O}(p^{-2}), \end{aligned} \quad (10.85)$$

$$\begin{aligned} e^p \cdot k_{3L} = e^p \cdot k_{3R} &= \frac{1}{M_2} \left(\sqrt{q^2 + M_3^2} \sqrt{p^2 + M_2^2} - pq \cos \phi \right) \\ &= \frac{pq}{M_2} \left[1 - \cos \phi + \frac{M_2^2}{2p^2} + \frac{M_3^2}{2q^2} \right] + \mathcal{O}(p^{-2}), \end{aligned} \quad (10.86)$$

$$\begin{aligned} e^L \cdot k_{1L} = e^L \cdot k_{1R} &= -\frac{p}{M_2} \left(\sqrt{p^2 + M_1^2} + \sqrt{p^2 + M_2^2} \right) \\ &= -\frac{p^2}{M_2} \left(2 + \frac{M_1^2}{2p^2} + \frac{M_2^2}{2p^2} \right) + \mathcal{O}(p^{-2}), \end{aligned} \quad (10.87)$$

$$\begin{aligned} e^L \cdot k_{3L} = e^L \cdot k_{3R} &= \frac{1}{M_2} \left(p \sqrt{q^2 + M_3^2} - q \sqrt{p^2 + M_2^2} \cos \phi \right) \\ &= \frac{pq}{M_2} \left[1 + \frac{M_3^2}{2q^2} - \left(1 + \frac{M_2^2}{2p^2} \right) \cos \phi \right] + \mathcal{O}(p^{-2}), \end{aligned} \quad (10.88)$$

$$e^T \cdot k_{1L} = e^T \cdot k_{1R} = 0, \quad (10.89)$$

$$e^T \cdot k_{3L} = e^T \cdot k_{3R} = -q \sin \phi, \quad (10.90)$$

which will be useful in the calculations of high energy string scattering amplitudes.

For the noncompactified open string, it was shown that [30, 31, 44], at each fixed mass level $M_{op}^2 = 2(N - 1)$, a four-point function is at the leading order in high energy (GR) only for states of the following form

$$|N, 2l, q\rangle \equiv (\alpha_{-1}^T)^{N-2l-2q} (\alpha_{-1}^L)^{2l} (\alpha_{-2}^L)^q |0, k\rangle \quad (10.91)$$

where $N \geq 2l + 2q, l, q \geq 0$. To avoid the complicated subleading order calculation due to the α_{-1}^L operator, we will choose the simple case $l = 0$. We made a similar choice when dealing with the high energy string/D-brane scatterings [41, 45, 126]. There is still one complication in the case of compactified string due to the possible choices of α_{-n}^{25} and $\tilde{\alpha}_{-m}^{25}$ in the vertex operator. However, it can be easily shown that for each fixed mass level with given quantized and winding momenta $(\frac{m}{R}, \frac{1}{2}nR)$, and thus fixed $N_R + N_L$ level, vertex operators containing α_{-n}^{25} or $\tilde{\alpha}_{-m}^{25}$ are subleading order in energy in the high energy expansion compared to other choices $\alpha_{-1}^T (\tilde{\alpha}_{-1}^T)$ and $\alpha_{-2}^L (\tilde{\alpha}_{-2}^L)$ on the noncompact directions. In conclusion, in the calculation of compactified closed string in the GR, we are going to consider tensor state of the form

$$|N_{L,R}, q_{L,R}\rangle \equiv (\alpha_{-1}^T)^{N_L-2q_L} (\alpha_{-2}^L)^{q_L} \otimes (\tilde{\alpha}_{-1}^T)^{N_R-2q_R} (\tilde{\alpha}_{-2}^L)^{q_R} |0\rangle, \quad (10.92)$$

at general $N_R + N_L$ level scattered from three other tachyon states with $N_R + N_L = 0$.

Note that, in the GR, one can identify e^p with e^L as usual [26–28]. However, in the MR, one can not identify e^p with e^L . This can be seen from Eq.(10.85) to Eq.(10.88). In the MR, instead of using the tensor vertex in Eq.(10.92), we will use

$$|N_{L,R}, q_{L,R}\rangle \equiv (\alpha_{-1}^T)^{N_L-2q_L} (\alpha_{-2}^P)^{q_L} \otimes (\tilde{\alpha}_{-1}^T)^{N_R-2q_R} (\tilde{\alpha}_{-2}^P)^{q_R} |0\rangle, \quad (10.93)$$

as the second vertex operator in the calculation of high energy scattering amplitudes. Note also that, in the MR, states in Eq.(10.93) may not be the only states which contribute to the high energy scattering amplitudes as in the GR. However, we will just choose these states to calculate the scattering amplitudes in order to compare with the corresponding high energy scattering amplitudes in the GR.

The high energy scattering amplitudes in the MR can be calculated to be

$$\begin{aligned}
A &= \varepsilon_{T^{N_L-2q_L}P^{q_L}, T^{N_R-2q_R}P^{q_R}} \int d^2 z_1 d^2 z_2 d^2 z_3 d^2 z_4 \\
&\cdot \left\langle V_1(z_1, \bar{z}_1) V_2^{T^{N_L-2q_L}P^{q_L}, T^{N_R-2q_R}P^{q_R}}(z_2, \bar{z}_2) V_3(z_3, \bar{z}_3) V_4(z_4, \bar{z}_4) \right\rangle \\
&= \varepsilon_{T^{N_L-2q_L}P^{q_L}, T^{N_R-2q_R}P^{q_R}} \int d^2 z_1 d^2 z_2 d^2 z_3 d^2 z_4 \left\langle e^{ik_{1L}X}(z_1) e^{ik_{1R}\tilde{X}}(\bar{z}_1) \right. \\
&\cdot (\partial X^T)^{N_L-2q_L} (i\partial^2 X^P)^{q_L} e^{ik_{2L}X}(z_2) (\bar{\partial}\tilde{X}^T)^{N_R-2q_R} (i\bar{\partial}^2 \tilde{X}^P)^{q_R} e^{ik_{2R}\tilde{X}}(\bar{z}_2) \\
&\cdot e^{ik_{3L}X}(z_3) e^{ik_{3R}\tilde{X}}(\bar{z}_3) e^{ik_{4L}X}(z_4) e^{ik_{4R}\tilde{X}}(\bar{z}_4) \left. \right\rangle \\
&= \int d^2 z_1 d^2 z_2 d^2 z_3 d^2 z_4 \cdot \left[\prod_{i<j} (z_i - z_j)^{k_{iL} \cdot k_{jL}} (\bar{z}_i - \bar{z}_j)^{k_{iR} \cdot k_{jR}} \right] \\
&\cdot \left[\frac{ie^T \cdot k_{1L}}{z_1 - z_2} + \frac{ie^T \cdot k_{3L}}{z_3 - z_2} + \frac{ie^T \cdot k_{4L}}{z_4 - z_2} \right]^{N_L-2q_L} \cdot \left[\frac{e^p \cdot k_{1L}}{(z_1 - z_2)^2} + \frac{e^p \cdot k_{3L}}{(z_3 - z_2)^2} + \frac{e^p \cdot k_{4L}}{(z_4 - z_2)^2} \right]^{q_L} \\
&\cdot \left[\frac{ie^T \cdot k_{1R}}{\bar{z}_1 - \bar{z}_2} + \frac{ie^T \cdot k_{3R}}{\bar{z}_3 - \bar{z}_2} + \frac{ie^T \cdot k_{4R}}{\bar{z}_4 - \bar{z}_2} \right]^{N_R-2q_R} \cdot \left[\frac{e^p \cdot k_{1R}}{(\bar{z}_1 - \bar{z}_2)^2} + \frac{e^p \cdot k_{3R}}{(\bar{z}_3 - \bar{z}_2)^2} + \frac{e^p \cdot k_{4R}}{(\bar{z}_4 - \bar{z}_2)^2} \right]^{q_R}.
\end{aligned} \tag{10.94}$$

After the standard $SL(2, C)$ gauge fixing, we get

$$\begin{aligned}
A &\simeq (-1)^{k_{1L} \cdot k_{2L} + k_{1R} \cdot k_{2R} + k_{1L} \cdot k_{3L} + k_{1R} \cdot k_{3R} + k_{2L} \cdot k_{3L} + k_{2R} \cdot k_{3R}} \\
&\cdot \int d^2 z \cdot z^{k_{1L} \cdot k_{2L}} \cdot \bar{z}^{k_{1R} \cdot k_{2R}} \cdot (1 - z)^{k_{2L} \cdot k_{3L}} (1 - \bar{z})^{k_{2R} \cdot k_{3R}} \\
&\cdot \left[\frac{ie^T \cdot k_{1L}}{z} - \frac{ie^T \cdot k_{3L}}{1 - z} \right]^{N_L-2q_L} \cdot \left[\frac{ie^T \cdot k_{1R}}{\bar{z}} - \frac{ie^T \cdot k_{3R}}{1 - \bar{z}} \right]^{N_R-2q_R} \\
&\cdot \left[\frac{e^p \cdot k_{1L}}{z^2} + \frac{e^p \cdot k_{3L}}{(1 - z)^2} \right]^{q_L} \cdot \left[\frac{e^p \cdot k_{1R}}{\bar{z}^2} + \frac{e^p \cdot k_{3R}}{(1 - \bar{z})^2} \right]^{q_R}.
\end{aligned} \tag{10.95}$$

By using Eqs.(10.85) to (10.90), the amplitude can be written as

$$\begin{aligned}
A &\sim (-1)^{n+q+q'+k_{1L} \cdot k_{2L} + k_{1R} \cdot k_{2R} + k_{1L} \cdot k_{3L} + k_{1R} \cdot k_{3R} + k_{2L} \cdot k_{3L} + k_{2R} \cdot k_{3R}} (q \sin \phi)^{N_L+N_R-2q_L-2q_R} \\
&\cdot \int d^2 z \cdot z^{k_{1L} \cdot k_{2L}} \cdot \bar{z}^{k_{1R} \cdot k_{2R}} \cdot (1 - z)^{k_{2L} \cdot k_{3L}} (1 - \bar{z})^{k_{2R} \cdot k_{3R}} \cdot \left[\frac{1}{1 - z} \right]^{N_L-2q_L} \left[\frac{1}{1 - \bar{z}} \right]^{N_R-2q_R} \\
&\cdot \left[\frac{-\frac{1}{M_2} \left(\sqrt{p^2 + M_1^2} \sqrt{p^2 + M_2^2} + p^2 \right)}{z^2} + \frac{\frac{1}{M_2} \left(\sqrt{q^2 + M_3^2} \sqrt{p^2 + M_2^2} - pq \cos \phi \right)}{(1 - z)^2} \right]^{q_L} \\
&\cdot \left[\frac{-\frac{1}{M_2} \left(\sqrt{p^2 + M_1^2} \sqrt{p^2 + M_2^2} + p^2 \right)}{\bar{z}^2} + \frac{\frac{1}{M_2} \left(\sqrt{q^2 + M_3^2} \sqrt{p^2 + M_2^2} - pq \cos \phi \right)}{(1 - \bar{z})^2} \right]^{q_R}
\end{aligned}$$

$$\begin{aligned}
&= (-1)^{k_{1L} \cdot k_{2L} + k_{1R} \cdot k_{2R} + k_{1L} \cdot k_{3L} + k_{1R} \cdot k_{3R} + k_{2L} \cdot k_{3L} + k_{2R} \cdot k_{3R}} (q \sin \phi)^{N_L + N_R} \left(\frac{1}{M_2 q^2 \sin^2 \phi} \right)^{q_L + q_R} \\
&\cdot \int d^2 z \cdot z^{k_{1L} \cdot k_{2L}} \cdot \bar{z}^{k_{1R} \cdot k_{2R}} \cdot (1 - z)^{k_{2L} \cdot k_{3L}} (1 - \bar{z})^{k_{2R} \cdot k_{3R}} \cdot \left[\frac{1}{1 - z} \right]^{N_L - 2q_L} \left[\frac{1}{1 - \bar{z}} \right]^{N_R - 2q_R} \\
&\cdot \sum_{i=0}^q \sum_{j=0}^{q'} \binom{q}{i} \binom{q'}{j} \left(\frac{\sqrt{p^2 + M_1^2} \sqrt{p^2 + M_2^2} + p^2}{z^2} \right)^i \left(\frac{\sqrt{p^2 + M_1^2} \sqrt{p^2 + M_2^2} + p^2}{\bar{z}^2} \right)^j \\
&= (-1)^{k_{1L} \cdot k_{2L} + k_{1R} \cdot k_{2R} + k_{1L} \cdot k_{3L} + k_{1R} \cdot k_{3R} + k_{2L} \cdot k_{3L} + k_{2R} \cdot k_{3R}} (q \sin \phi)^{N_L + N_R} \\
&\cdot \left(-\frac{\sqrt{q^2 + M_3^2} \sqrt{p^2 + M_2^2} - pq \cos \phi}{M_2 q^2 \sin^2 \phi} \right)^{q_L + q_R} \\
&\cdot \sum_{i=0}^{q_L} \sum_{j=0}^{q_R} \binom{q_L}{i} \binom{q_R}{j} \left(\frac{\sqrt{p^2 + M_1^2} \sqrt{p^2 + M_2^2} + p^2}{-\sqrt{q^2 + M_3^2} \sqrt{p^2 + M_2^2} + pq \cos \phi} \right)^{i+j} \\
&\cdot \frac{\sin[-\pi(k_{1R} \cdot k_{2R} - 2j)] \sin[-\pi(k_{2R} \cdot k_{3R} - N_R + 2j)]}{\sin[-\pi(1 + k_{1R} \cdot k_{2R} + k_{2R} \cdot k_{3R} - N_R)]} \\
&\cdot \frac{\Gamma(1 + k_{1R} \cdot k_{2R} - 2j) \Gamma(1 + k_{2R} \cdot k_{3R} - N_R + 2j)}{\Gamma(2 + k_{1R} \cdot k_{2R} + k_{2R} \cdot k_{3R} - N_R)} \\
&\cdot \frac{\Gamma(1 + k_{1L} \cdot k_{2L} - 2i) \Gamma(1 + k_{2L} \cdot k_{3L} + 2i - N_L)}{\Gamma(2 + k_{1L} \cdot k_{2L} + k_{2L} \cdot k_{3L} - N_L)} \tag{10.96}
\end{aligned}$$

where, as in the calculation of section B.2 for the GR, we have used Eq.(10.78) to do the integration. It is easy to do the following approximations for the gamma functions

$$\begin{aligned}
A &\simeq (-1)^{k_{1L} \cdot k_{2L} + k_{1R} \cdot k_{2R} + k_{1L} \cdot k_{3L} + k_{1R} \cdot k_{3R} + k_{2L} \cdot k_{3L} + k_{2R} \cdot k_{3R}} (q \sin \phi)^{N_L + N_R} \\
&\cdot \left(-\frac{\sqrt{q^2 + M_3^2} \sqrt{p^2 + M_2^2} - pq \cos \phi}{M_2 q^2 \sin^2 \phi} \right)^{q_L + q_R} \\
&\cdot \sum_{i=0}^{q_L} \sum_{j=0}^{q_R} \binom{q_L}{i} \binom{q_R}{j} \left(\frac{\sqrt{p^2 + M_1^2} \sqrt{p^2 + M_2^2} + p^2}{-\sqrt{q^2 + M_3^2} \sqrt{p^2 + M_2^2} + pq \cos \phi} \right)^{i+j} \\
&\cdot \frac{\sin[-\pi k_{1R} \cdot k_{2R}] \sin[-\pi k_{2R} \cdot k_{3R}]}{\sin[-\pi(1 + k_{1R} \cdot k_{2R} + k_{2R} \cdot k_{3R})]} \\
&\cdot \frac{\Gamma(1 + k_{1R} \cdot k_{2R}) \Gamma(1 + k_{2R} \cdot k_{3R}) \Gamma(1 + k_{1L} \cdot k_{2L}) \Gamma(1 + k_{2L} \cdot k_{3L})}{\Gamma(2 + k_{1R} \cdot k_{2R} + k_{2R} \cdot k_{3R}) \Gamma(2 + k_{1L} \cdot k_{2L} + k_{2L} \cdot k_{3L})} \\
&\cdot \frac{(k_{1R} \cdot k_{2R})^{-2j} (k_{2R} \cdot k_{3R})^{-N_R + 2j} (k_{1L} \cdot k_{2L})^{-2i} (k_{2L} \cdot k_{3L})^{-N_L + 2i}}{(k_{1R} \cdot k_{2R} + k_{2R} \cdot k_{3R})^{-N_R} (k_{1L} \cdot k_{2L} + k_{2L} \cdot k_{3L})^{-N_L}}. \tag{10.97}
\end{aligned}$$

One can now do the double summation and drop out the $M_{iL,R}$ terms to get

$$\begin{aligned}
A \simeq & \left(-\frac{q \sin \phi (s_L + t_L)}{t_L} \right)^{N_L} \left(-\frac{q \sin \phi (s_R + t_R)}{t_R} \right)^{N_R} \left(\frac{1}{2M_2 q^2 \sin^2 \phi} \right)^{q_L + q_R} \\
& \cdot \left(\left(t_R - 2\vec{K}_{2R} \cdot \vec{K}_{3R} \right) + \frac{t_R^2 (s_R - 2\vec{K}_{1R} \cdot \vec{K}_{2R})}{s_R^2} \right)^{q_R} \\
& \cdot \left(\left(t_L - 2\vec{K}_{2L} \cdot \vec{K}_{3L} \right) + \frac{t_L^2 (s_L - 2\vec{K}_{1L} \cdot \vec{K}_{2L})}{s_L^2} \right)^{q_L} \\
& \cdot \frac{\sin(\pi s_L/2) \sin(\pi t_R/2)}{\sin(\pi u_L/2)} B\left(-1 - \frac{t_R}{2}, -1 - \frac{u_R}{2}\right) B\left(-1 - \frac{t_L}{2}, -1 - \frac{u_L}{2}\right). \quad (10.98)
\end{aligned}$$

Eq.(10.98) is valid for $E^2 \gg N_R + N_L$, $M_2^2 \gg N_R + N_L$.

4. The infinite linear relations in the GR

For the special case of GR with $E^2 \gg M_2^2$, one can identify q with p , and the amplitude in Eq.(10.98) further reduces to

$$\begin{aligned}
\lim_{E^2 \gg M_2^2} A = & \left(\frac{2p \cos^3 \frac{\phi}{2}}{\sin \frac{\phi}{2}} \right)^{N_L + N_R} \left(-\frac{1}{2M_2} \right)^{q_L + q_R} \frac{\sin(\pi s_L/2) \sin(\pi t_R/2)}{\sin(\pi u_L/2)} \\
& \cdot B\left(-1 - \frac{t_R}{2}, -1 - \frac{u_R}{2}\right) B\left(-1 - \frac{t_L}{2}, -1 - \frac{u_L}{2}\right). \quad (10.99)
\end{aligned}$$

It is crucial to note that the high energy limit of the beta function with $s + t + u = 2n - 8$ is [26, 27]

$$\begin{aligned}
B\left(-1 - \frac{t}{2}, -1 - \frac{u}{2}\right) &= \frac{\Gamma(-\frac{t}{2} - 1) \Gamma(-\frac{u}{2} - 1)}{\Gamma(\frac{s}{2} + 2)} \\
&\simeq E^{-1-2n} \left(\sin \frac{\phi}{2} \right)^{-3} \left(\cos \frac{\phi}{2} \right)^{5-2n} \\
&\cdot \exp\left(-\frac{t \ln t + u \ln u - (t + u) \ln(t + u)}{2}\right) \quad (10.100)
\end{aligned}$$

where we have calculated the approximation up to the next leading order in energy E . Note the appearance of the prepower factors in front of the exponential fall-off factor. For our purpose here, with Eq.(10.74), we have

$$s_{L,R} + t_{L,R} + u_{L,R} = \sum_i M_{iL,R}^2 = 2N_{L,R} - 8, \quad (10.101)$$

and the high energy limit of the beta functions in Eq.(10.28) can be further calculated to be

$$\begin{aligned}
& B\left(-1 - \frac{t_R}{2}, -1 - \frac{u_R}{2}\right) B\left(-1 - \frac{t_L}{2}, -1 - \frac{u_L}{2}\right) \\
& \simeq E^{-1-2(N_L+N_R)} \left(\sin \frac{\phi}{2}\right)^{-3} \left(\cos \frac{\phi}{2}\right)^{5-2(N_L+N_R)} \\
& \cdot \exp\left(-\frac{t_L \ln t_L + u_L \ln u_L - (t_L + u_L) \ln(t_L + u_L)}{2}\right) \\
& \cdot \exp\left(-\frac{t_R \ln t_R + u_R \ln u_R - (t_R + u_R) \ln(t_R + u_R)}{2}\right) \\
& \simeq E^{-1-2(N_L+N_R)} \left(\sin \frac{\phi}{2}\right)^{-3} \left(\cos \frac{\phi}{2}\right)^{5-2(N_L+N_R)} \exp\left(-\frac{t \ln t + u \ln u - (t + u) \ln(t + u)}{4}\right)
\end{aligned} \tag{10.102}$$

where we have implicitly used the relation $\alpha'_{\text{closed}} = 4\alpha'_{\text{open}} = 2$. By combining Eq.(10.99) and Eq.(10.102), we end up with

$$\begin{aligned}
\lim_{E^2 \gg M_2^2} A & \simeq \left(-\frac{2 \cot \frac{\phi}{2}}{E}\right)^{N_L+N_R} \left(-\frac{1}{2M_2}\right)^{q_L+q_R} E^{-1} \left(\sin \frac{\phi}{2}\right)^{-3} \left(\cos \frac{\phi}{2}\right)^5 \\
& \cdot \frac{\sin(\pi s_L/2) \sin(\pi t_R/2)}{\sin(\pi u_L/2)} \exp\left(-\frac{t \ln t + u \ln u - (t + u) \ln(t + u)}{4}\right).
\end{aligned} \tag{10.103}$$

We see that there is a $(\frac{m}{R}, \frac{1}{2}nR)$ dependence in the $\frac{\sin(\pi s_L/2) \sin(\pi t_R/2)}{\sin(\pi u_L/2)}$ factor in our final result. This is physically consistent as one expects a $(\frac{m}{R}, \frac{1}{2}nR)$ dependent Regge-pole and zero structures in the high energy string scattering amplitudes.

In conclusion, in the GR, for each fixed mass level with given quantized and winding momenta $(\frac{m}{R}, \frac{1}{2}nR)$ (thus fixed N_L and N_R by Eq.(10.53)), we have obtained infinite linear relations among high energy scattering amplitudes of different string states with various (q_L, q_R) . Note also that this result reproduces the correct ratios $\left(-\frac{1}{2M_2}\right)^{q_L+q_R}$ obtained in the previous works [41, 45, 126]. However, the mass parameter M_2 here depends on $(\frac{m}{R}, \frac{1}{2}nR)$. It is also interesting to see that, if not for the $(\frac{m}{R}, \frac{1}{2}nR)$ dependence in the $\frac{\sin(\pi s_L/2) \sin(\pi t_R/2)}{\sin(\pi u_L/2)}$ factor in the high energy scattering amplitudes in the GR, we would have had a linear relation among scattering amplitudes of different string states in different mass levels with fixed $(N_R + N_L)$.

Presumably, the infinite linear relations obtained above can be reproduced by using the method of high energy ZNS, or high energy Ward identities, adopted in the previous chapters [26–31, 33, 35, 41, 44, 126]. The existence of Soliton ZNS at arbitrary mass levels was

constructed in chapter IV [6]. A closer look in this direction seems worthwhile. In the chapter, however, we are more interested in understanding the power-law behavior of the high energy string scattering amplitudes and breakdown of the infinite linear relations as we will discuss in the next section.

5. Power-law and breakdown of the infinite linear relations in the MR

In this section we discuss the power-law behavior of high energy string scattering amplitudes in a compact space. We will see that, in the MR, the infinite linear relations derived in section B.4 *break down* and, simultaneously, the UV exponential fall-off behavior of high energy string scattering amplitudes *enhances* to power-law behavior. The power-law behavior of high energy string scatterings in a compact space was first suggested by Mende [53]. Here we give a mathematically more concrete description. It is easy to see that the "power law" condition, i.e. Eq.(3.7) in Mende's paper [53],

$$k_{1L} \cdot k_{2L} + k_{1R} \cdot k_{2R} = \text{constant}, \quad (10.104)$$

turns out to be

$$\begin{aligned} & - (k_{1L} \cdot k_{2L} + k_{1R} \cdot k_{2R}) \\ &= \sqrt{p^2 + M_1^2} \cdot \sqrt{p^2 + M_2^2} + p^2 + \left(\vec{K}_{1L} \cdot \vec{K}_{2L} + \vec{K}_{1R} \cdot \vec{K}_{2R} \right) \\ &= \sqrt{p^2 + M_1^2} \cdot \sqrt{p^2 + M_2^2} + p^2 + 2 \left(\vec{K}_1 \cdot \vec{K}_2 + \vec{L}_1 \cdot \vec{L}_2 \right) \\ &= \text{constant}. \end{aligned} \quad (10.105)$$

The condition to achieve power-law behavior for the compactified open string scatterings, Eq.(10.36), is replaced by

$$s_L = \text{constant}, s_R = \text{constant} \quad (10.106)$$

It is easy to see that the condition, Eq.(10.106), leads to the power-law behavior of the compactified closed string scattering amplitudes. To satisfy the condition in Eq.(10.106), we define the following "super-highly" winding kinematic regime

$$n_i^2 \gg p^2 \simeq q^2 \gg N_R + N_L. \quad (10.107)$$

Note that all m_i were chosen to vanish in order to satisfy the conservations of compactified momentum and winding number respectively [54]. For the choice of the kinematic regime in Eq.(10.107), Eq.(10.65) and Eq.(10.106) imply

$$\lim_{p \rightarrow \infty} \frac{\sqrt{p^2 + M_1^2} \cdot \sqrt{p^2 + M_2^2} + p^2}{2(K_1 K_2 + L_1 L_2)} = \lim_{p \rightarrow \infty} \frac{\sqrt{p^2 + M_1^2} \cdot \sqrt{p^2 + M_2^2} + p^2}{2 \left(\frac{m_1 m_2}{R^2} + \frac{1}{4} n_1 n_2 R^2 \right)} = -1. \quad (10.108)$$

Note that since we have set $m_i = 0$, Eq.(10.108) is similar to Eq.(10.37) for the compactified open string case, and one can get nontrivial solution for Eq.(10.39) with signs of $\lambda_1 = \frac{2p}{n_1 R}$ and $\lambda_2 = -\frac{2p}{n_2 R}$ the same. This completes the discussion of power-law regime at fixed angle for high energy compactified closed string scatterings. The "super-highly" winding regime derived in this section is to correct the "Mende regime"

$$E^2 \simeq M^2 \gg N_R + N_L \quad (10.109)$$

discussed in [54]. The regime defined in Eq.(10.109) is indeed exponential fall-off behaved rather than power-law claimed in [54].

Part III

Stringy symmetries of Regge string scattering amplitudes

In this part of the review, we are going to discuss string scatterings in the high energy, fixed momentum transfer regime or Regge regime (RR) [56–61]. See also [141–143]. We will see that Regge string scattering amplitudes contain information of the theory in complementary to string scattering amplitudes in the high energy, fixed angle or Gross regime (GR). The UV behavior of high energy string scatterings in the GR (hard string scatterings) is well known to be very soft exponential fall-off, while that of RR is power-law. There are some other fundamental differences and links between the calculations of Regge string scatterings and hard string scatterings, which we list below :

A. The number of high energy scattering amplitudes for each fixed mass level in the RR is much more numerous than that of GR calculated previously [63].

B. The saddle-point method is not applicable in the calculation of Regge string scattering amplitudes. However, a direct calculation is manageable and one finds that all string amplitudes in the RR can be expressed in terms of a finite sum of Kummer functions of the second kind [63, 68–70].

C. There is an interesting link between scattering amplitudes of the RR and GR. By proving a set of new Stirling number identities [66], one can reproduce from amplitudes in the RR the ratios calculated in the GR [63, 64].

D. For the high energy scattering amplitudes in the fixed angle regime, one has the identification $e^P = e^L$, while in the Regge regime $e^P \neq e^L$ [63].

E. The decoupling of ZNS applies to all kinematic regimes including the RR and the GR. However, the linear relations obtained from decoupling of ZNS in the RR are not good enough to solve all the amplitudes in the RR at each fixed mass level. Instead, the recurrence relations among RR amplitudes obtained from Kummer functions can be used to fix all RR amplitudes in terms of one amplitude [70].

F. All the RR amplitudes can be expressed in terms of one single Appell function F_1 [73]. This result enables us to derive infinite number of recurrence relations among RR amplitudes at arbitrary mass levels, which are conjectured to be related to the known $SL(5, C)$ dynamical symmetry of F_1 .

The part III of the paper is organized as following. In chapter XI we calculate Regge string scattering amplitudes in terms of Kummer functions [63]. We then prove a set of Stirling number identities [66] and use them to reproduce ratios among hard string scattering amplitudes in the GR discussed in chapter V. Finally we calculate recurrence relations among Regge string scattering amplitudes, and use them to prove Regge stringy Ward identities or decoupling of ZNS in the RR for the first few mass levels [70]. In Chapter XII, we generalize the calculations to four classes of Regge superstring scattering amplitudes and reproduce the ratios calculated in chapter VIII. In addition, discover new high energy superstring scattering amplitudes with polarizations *orthogonal* to the scattering plane [64].

In Chapter XIII we generalize the calculation of four tachyon BPST vertex operator [61] to the general high spin cases [72], and derive the recurrence relations among these higher spin BPST vertex operators. In Chapter XIV we discuss higher spin Regge string states scattered from D-particle [42]. We will obtain the complete GR ratios, which include a subset calculated in section IX.A.3, from Regge string states scattered from D-particle.

In addition, we will see that although there is no factorization for closed string D-particle scattering amplitudes into two channels of open string D-particle scattering amplitudes, the complete ratios are *factorized* which came as a surprise.

In chapter XV we discover that all RR amplitudes can be expressed in terms of one single Appell function F_1 [73]. More general recurrence relations among RR amplitudes will be derived. We will also discuss the $SL(5, C)$ dynamical symmetry of F_1 which is argued to be closely related to high energy spacetime symmetry of $26D$ bosonic string theory.

XI. KUMMER FUNCTIONS U AND PATTERNS OF REGGE STRING SCATTERING AMPLITUDES (RSSA)

In this chapter, we first calculate a subclass of Regge string scattering amplitudes (RSSA) and expressed them in terms of Kummer functions of the second kind [68, 69]. We then prove in section B a set of Stirling number identities [66] and use them to reproduce ratios among hard string scattering amplitudes calculated in chapter V. In section C, we show that all RSSA are power law behaved, and the exponents of the power law are universal and are independent of the mass levels [63]. In section D, we calculate the most general RSSA and derive recurrence relations among them [70]. We show that, for the first few mass levels, the decoupling of ZNS in the RR or Regge stringy Ward identities can be derived from these recurrence relations [70]. This shows that, in contrast to the GR considered in chapter V, recurrence relations are more fundamental than Regge stringy Ward identities. Finally we prove that all RSSA can be solved by these recurrence relations and expressed in terms of one single RSSA [70].

A. Kummer functions and RSSA

We now begin to discuss high energy string scatterings in the RR. That is in the kinematic regime

$$s \rightarrow \infty, \sqrt{-t} = \text{fixed (but } \sqrt{-t} \neq \infty). \quad (11.1)$$

As in the case of GR, we only need to consider the polarizations on the scattering plane, which is defined in Appendix C. Appendix C also includes the kinematic set up. Instead of using (E, θ) as the two independent kinematic variables in the GR, we choose to use (s, t)

in the RR. One of the reason has been, in the RR, $t \sim E\theta$ is fixed, and it is more convenient to use (s, t) rather than (E, θ) . In the RR, to the lowest order, Eqs.(C.13) to (C.18) reduce to

$$e^P \cdot k_1 = -\frac{1}{M_2} \left(\sqrt{p^2 + M_1^2} \sqrt{p^2 + M_2^2} + p^2 \right) \simeq -\frac{s}{2M_2}, \quad (11.2a)$$

$$e^L \cdot k_1 = -\frac{p}{M_2} \left(\sqrt{p^2 + M_1^2} + \sqrt{p^2 + M_2^2} \right) \simeq -\frac{s}{2M_2}, \quad (11.2b)$$

$$e^T \cdot k_1 = 0 \quad (11.2c)$$

and

$$e^P \cdot k_3 = \frac{1}{M_2} \left(\sqrt{q^2 + M_3^2} \sqrt{p^2 + M_2^2} - pq \cos \theta \right) \simeq -\frac{\tilde{t}}{2M_2} \equiv -\frac{t - M_2^2 - M_3^2}{2M_2}, \quad (11.3a)$$

$$e^L \cdot k_3 = \frac{1}{M_2} \left(p \sqrt{q^2 + M_3^2} - q \sqrt{p^2 + M_2^2} \cos \theta \right) \simeq -\frac{\tilde{t}'}{2M_2} \equiv -\frac{t + M_2^2 - M_3^2}{2M_2}, \quad (11.3b)$$

$$e^T \cdot k_3 = -q \sin \phi \simeq -\sqrt{-t}. \quad (11.3c)$$

Note that e^P *does not* approach to e^L in the RR. This is very different from the case of GR. In the following discussion, we will calculate the amplitudes for the longitudinal polarization e^L . For the e^P amplitudes, the results can be trivially modified. We will find that the number of high energy scattering amplitudes for each fixed mass level in the RR is much more numerous than that of GR calculated previously. On the other hand, it seems that the saddle-point method used in the GR is not applicable in the RR. We will first calculate the string scattering amplitudes on the scattering plane (e^L, e^T) for the mass level $M_2^2 = 4$. In the mass level $M_2^2 = 4$ ($M_1^2 = M_3^2 = M_4^2 = -2$), it turns out that there are eight high energy amplitudes in the RR

$$\begin{aligned} & \alpha_{-1}^T \alpha_{-1}^T \alpha_{-1}^T |0\rangle, \alpha_{-1}^L \alpha_{-1}^T \alpha_{-1}^T |0\rangle, \alpha_{-1}^L \alpha_{-1}^L \alpha_{-1}^T |0\rangle, \alpha_{-1}^L \alpha_{-1}^L \alpha_{-1}^L |0\rangle, \\ & \alpha_{-1}^T \alpha_{-2}^T |0\rangle, \alpha_{-1}^T \alpha_{-2}^L |0\rangle, \alpha_{-1}^L \alpha_{-2}^T |0\rangle, \alpha_{-1}^L \alpha_{-2}^L |0\rangle. \end{aligned} \quad (11.4)$$

The $s - t$ channel of these amplitudes can be calculated to be

$$\begin{aligned} A^{TTT} &= \int_0^1 dx \cdot x^{k_1 \cdot k_2} (1-x)^{k_2 \cdot k_3} \cdot \left(\frac{ie^T \cdot k_1}{x} - \frac{ie^T \cdot k_3}{1-x} \right)^3 \\ &\simeq -i (\sqrt{-t})^3 \frac{\Gamma(-\frac{s}{2} - 1) \Gamma(-\frac{\tilde{t}}{2} - 1)}{\Gamma(\frac{u}{2} + 3)} \cdot \left(-\frac{1}{8} s^3 + \frac{1}{2} s \right), \end{aligned} \quad (11.5)$$

$$\begin{aligned}
A^{LTT} &= \int_0^1 dx \cdot x^{k_1 \cdot k_2} (1-x)^{k_2 \cdot k_3} \cdot \left(\frac{ie^T \cdot k_1}{x} - \frac{ie^T \cdot k_3}{1-x} \right)^2 \left(\frac{ie^L \cdot k_1}{x} - \frac{ie^L \cdot k_3}{1-x} \right) \\
&\simeq -i (\sqrt{-t})^2 \left(-\frac{1}{2M_2} \right) \frac{\Gamma(-\frac{s}{2}-1) \Gamma(-\frac{\bar{t}}{2}-1)}{\Gamma(\frac{u}{2}+3)} \cdot \left[\frac{3}{4}s^3 - \frac{t}{4}s^2 - \left(\frac{t}{2} + 3 \right) s \right], \quad (11.6)
\end{aligned}$$

$$\begin{aligned}
A^{LLT} &= \int_0^1 dx \cdot x^{k_1 \cdot k_2} (1-x)^{k_2 \cdot k_3} \cdot \left(\frac{ie^T \cdot k_1}{x} - \frac{ie^T \cdot k_3}{1-x} \right) \left(\frac{ie^L \cdot k_1}{x} - \frac{ie^L \cdot k_3}{1-x} \right)^2 \\
&\simeq -i (\sqrt{-t}) \left(-\frac{1}{2M_2} \right)^2 \frac{\Gamma(-\frac{s}{2}-1) \Gamma(-\frac{\bar{t}}{2}-1)}{\Gamma(\frac{u}{2}+3)} \\
&\cdot \left[\left(\frac{1}{4}t - \frac{9}{2} \right) s^3 + \left(\frac{1}{4}t^2 + \frac{7}{2}t \right) s^2 + \frac{(t+6)^2}{2}s \right], \quad (11.7)
\end{aligned}$$

$$\begin{aligned}
A^{LLL} &= \int_0^1 dx \cdot x^{k_1 \cdot k_2} (1-x)^{k_2 \cdot k_3} \cdot \left(\frac{ie^L \cdot k_1}{x} - \frac{ie^L \cdot k_3}{1-x} \right)^3 \\
&\simeq -i \left(-\frac{1}{2M_2} \right)^3 \frac{\Gamma(-\frac{s}{2}-1) \Gamma(-\frac{\bar{t}}{2}-1)}{\Gamma(\frac{u}{2}+3)} \\
&\cdot \left[-\left(\frac{11}{2}t - 27 \right) s^3 - 6(t^2 + 6t) s^2 - \frac{(t+6)^3}{2}s \right], \quad (11.8)
\end{aligned}$$

$$\begin{aligned}
A^{TT} &= \int_0^1 dx \cdot x^{k_1 \cdot k_2} (1-x)^{k_2 \cdot k_3} \cdot \left(\frac{ie^T \cdot k_1}{x} - \frac{ie^T \cdot k_3}{1-x} \right) \left[\frac{e^T \cdot k_1}{x^2} + \frac{e^T \cdot k_3}{(1-x)^2} \right] \\
&\simeq -i (\sqrt{-t})^2 \frac{\Gamma(-\frac{s}{2}-1) \Gamma(-\frac{\bar{t}}{2}-1)}{\Gamma(\frac{u}{2}+3)} \left(-\frac{1}{8}s^3 + \frac{1}{2}s \right), \quad (11.9)
\end{aligned}$$

$$\begin{aligned}
A^{TL} &= \int_0^1 dx \cdot x^{k_1 \cdot k_2} (1-x)^{k_2 \cdot k_3} \cdot \left(\frac{ie^T \cdot k_1}{x} - \frac{ie^T \cdot k_3}{1-x} \right) \left[\frac{e^L \cdot k_1}{x^2} + \frac{e^L \cdot k_3}{(1-x)^2} \right] \\
&\simeq i (\sqrt{-t}) \left(-\frac{1}{2M_2} \right) \frac{\Gamma(-\frac{s}{2}-1) \Gamma(-\frac{\bar{t}}{2}-1)}{\Gamma(\frac{u}{2}+3)} \\
&\cdot \left[-\left(\frac{1}{8}t + \frac{3}{4} \right) s^3 - \frac{1}{8}(t^2 - 2t) s^2 - \left(\frac{1}{4}t^2 - t - 3 \right) s \right], \quad (11.10)
\end{aligned}$$

$$\begin{aligned}
A^{LT} &= \int_0^1 dx \cdot x^{k_1 \cdot k_2} (1-x)^{k_2 \cdot k_3} \cdot \left(\frac{ie^L \cdot k_1}{x} - \frac{ie^L \cdot k_3}{1-x} \right) \left[\frac{e^T \cdot k_1}{x^2} + \frac{e^T \cdot k_3}{(1-x)^2} \right] \\
&\simeq i (\sqrt{-t}) \left(-\frac{1}{2M_2} \right) \frac{\Gamma(-\frac{s}{2}-1) \Gamma(-\frac{\bar{t}}{2}-1)}{\Gamma(\frac{u}{2}+3)} \cdot \left[\frac{3}{4}s^3 - \frac{t}{4}s^2 - \left(\frac{t}{2} + 3 \right) s \right], \quad (11.11)
\end{aligned}$$

and

$$\begin{aligned}
A^{LL} &= \int_0^1 dx \cdot x^{k_1 \cdot k_2} (1-x)^{k_2 \cdot k_3} \cdot \left(\frac{ie^L \cdot k_1}{x} - \frac{ie^L \cdot k_3}{1-x} \right) \left[\frac{e^L \cdot k_1}{x^2} + \frac{e^L \cdot k_3}{(1-x)^2} \right] \\
&\simeq i \left(-\frac{1}{2M_2} \right)^2 \frac{\Gamma\left(-\frac{s}{2}-1\right) \Gamma\left(-\frac{\tilde{t}}{2}-1\right)}{\Gamma\left(\frac{u}{2}+3\right)} \\
&\cdot \left[\left(\frac{3}{4}t + \frac{9}{2} \right) s^3 + (t^2 - 4t) s^2 + \left(\frac{1}{4}t^3 + \frac{1}{2}t^2 - 9t - 18 \right) s \right]. \tag{11.12}
\end{aligned}$$

From the above calculation, one can easily see that all the amplitudes are in the same leading order ($\sim s^3$) in the RR, while in the GR only A^{TTT} , A^{LLT} and A^{TL} are in the leading order ($\sim t^{3/2}s^3$ or $t^{5/2}s^2$), all other amplitudes are in the subleading orders. On the other hand, one notes that, for example, the term $\sim \sqrt{-t}t^2s^2$ in A^{LLT} and A^{TL} are in the leading order in the GR, but are in the subleading order in the RR. On the contrary, the terms $\sqrt{-t}s^3$ in A^{LLT} and A^{TL} are in the subleading order in the GR, but are in the leading order in the RR. These observations suggest that the high energy string scattering amplitudes in the GR and RR contain information complementary to each other.

One important observation for high energy amplitudes in the RR is for those amplitudes with the same structure as those of the GR in Eq.(5.67). For these amplitudes, the relative ratios of the coefficients of the highest power of t in the leading order amplitudes in the RR can be calculated to be

$$A^{TTT} = -i(\sqrt{-t}) \frac{\Gamma\left(-\frac{s}{2}-1\right) \Gamma\left(-\frac{\tilde{t}}{2}-1\right)}{\Gamma\left(\frac{u}{2}+3\right)} \cdot \left(\frac{1}{8}ts^3 \right) \sim \frac{1}{8}, \tag{11.13}$$

$$A^{LLT} = -i(\sqrt{-t}) \left(-\frac{1}{2M_2} \right)^2 \frac{\Gamma\left(-\frac{s}{2}-1\right) \Gamma\left(-\frac{\tilde{t}}{2}-1\right)}{\Gamma\left(\frac{u}{2}+3\right)} \left(\frac{1}{4}ts^3 \right) \sim \frac{1}{64}, \tag{11.14}$$

$$A^{TL} = i(\sqrt{-t}) \left(-\frac{1}{2M_2} \right) \frac{\Gamma\left(-\frac{s}{2}-1\right) \Gamma\left(-\frac{\tilde{t}}{2}-1\right)}{\Gamma\left(\frac{u}{2}+3\right)} \cdot \left(-\frac{1}{8}ts^3 \right) \sim -\frac{1}{32}, \tag{11.15}$$

which reproduces the ratios in the GR in Eq.(5.16). Note that the symmetrized and anti-symmetrized amplitudes are defined as

$$T^{(TL)} = \frac{1}{2} (T^{TL} + T^{LT}), \tag{11.16}$$

$$T^{[TL]} = \frac{1}{2} (T^{TL} - T^{LT}); \tag{11.17}$$

and similarly for the amplitudes $A^{(TL)}$ and $A^{[TL]}$ in the RR. Note that $T^{LT} \sim (\alpha_{-1}^L)(\alpha_{-2}^T)|0\rangle$ in the GR is of subleading order in energy, while A^{LT} in the RR is of leading order in energy.

However, the contribution of the amplitude A^{LT} to $A^{(TL)}$ and $A^{[TL]}$ in the RR will not affect the ratios calculated above. As we will see next, this interesting result can be generalized to all mass levels in the string spectrum.

We first calculate high energy string scattering amplitudes in the RR for the arbitrary mass levels. Instead of states in Eq.(5.67) for the GR, one can argue that the most general string states (ignore the e^P amplitudes) one needs to consider at each fixed mass level $N = \sum_{n,m} nk_n + mq_m$ for the RR are

$$|k_n, q_m\rangle = \prod_{n>0} (\alpha_{-n}^T)^{k_n} \prod_{m>0} (\alpha_{-m}^L)^{q_m} |0\rangle. \quad (11.18)$$

These RR amplitudes are good enough to reproduce the GR ratios calculated previously. By the simple kinematics $e^T \cdot k_1 = 0$, and the energy power counting of the string amplitudes, we end up with the following rules to simplify the calculation for the leading order amplitudes in the RR:

$$\alpha_{-n}^T : \quad 1 \text{ term (contraction of } ik_3 \cdot X \text{ with } \varepsilon_T \cdot \partial^n X), \quad (11.19)$$

$$\alpha_{-n}^L : \quad \begin{cases} n > 1, & 1 \text{ term} \\ n = 1 & 2 \text{ terms (contraction of } ik_1 \cdot X \text{ and } ik_3 \cdot X \text{ with } \varepsilon_L \cdot \partial^n X). \end{cases} \quad (11.20)$$

The $s - t$ channel scattering amplitudes of this state with three other tachyonic states can be calculated to be

$$\begin{aligned} A^{(k_n, q_m)} &= \int_0^1 dx x^{k_1 \cdot k_2} (1-x)^{k_2 \cdot k_3} \left[\frac{ie^L \cdot k_1}{-x} + \frac{ie^L \cdot k_3}{1-x} \right]^{q_1} \\ &\cdot \prod_{n=1} \left[\frac{ie^T \cdot k_3 (n-1)!}{(1-x)^n} \right]^{k_n} \prod_{m=2} \left[\frac{ie^L \cdot k_3 (m-1)!}{(1-x)^m} \right]^{q_m} \\ &= \left(\frac{-i\tilde{t}'}{2M_2} \right)^{q_1} \sum_{j=0}^{q_1} \binom{q_1}{j} \left(\frac{s}{-\tilde{t}} \right)^j \int_0^1 dx x^{k_1 \cdot k_2 - j} (1-x)^{k_2 \cdot k_3 + j - \sum_{n,m} (nk_n + mq_m)} \\ &\cdot \prod_{n=1} [i\sqrt{-t}(n-1)!]^{k_n} \prod_{m=2} \left[i\tilde{t}'(m-1)! \left(-\frac{1}{2M_2} \right) \right]^{q_m} \\ &= \left(\frac{-i\tilde{t}'}{2M_2} \right)^{q_1} \sum_{j=0}^{q_1} \binom{q_1}{j} \left(\frac{s}{-\tilde{t}} \right)^j B(k_1 \cdot k_2 - j + 1, k_2 \cdot k_3 + j - N + 1) \\ &\cdot \prod_{n=1} [i\sqrt{-t}(n-1)!]^{k_n} \prod_{m=2} \left[i\tilde{t}'(m-1)! \left(-\frac{1}{2M_2} \right) \right]^{q_m}. \end{aligned} \quad (11.21)$$

The Beta function above can be approximated in the large s , but fixed t limit as follows

$$\begin{aligned}
& B(k_1 \cdot k_2 - j + 1, k_2 \cdot k_3 + j - N + 1) \\
&= B\left(-1 - \frac{s}{2} + N - j, -1 - \frac{t}{2} + j\right) \\
&= \frac{\Gamma(-1 - \frac{s}{2} + N - j)\Gamma(-1 - \frac{t}{2} + j)}{\Gamma(\frac{u}{2} + 2)} \\
&\approx B\left(-1 - \frac{1}{2}s, -1 - \frac{t}{2}\right) \left(-1 - \frac{s}{2}\right)^{N-j} \left(\frac{u}{2} + 2\right)^{-N} \left(-1 - \frac{t}{2}\right)_j \\
&\approx B\left(-1 - \frac{1}{2}s, -1 - \frac{t}{2}\right) \left(-\frac{s}{2}\right)^{-j} \left(-1 - \frac{t}{2}\right)_j.
\end{aligned} \tag{11.22}$$

where

$$(a)_j = a(a+1)(a+2)\dots(a+j-1) \tag{11.23}$$

is the Pochhammer symbol. The leading order amplitude in the RR can then be written as

$$\begin{aligned}
A^{(k_n, q_m)} &= \left(\frac{-i\tilde{t}'}{2M_2}\right)^{q_1} B\left(-1 - \frac{1}{2}s, -1 - \frac{t}{2}\right) \sum_{j=0}^{q_1} \binom{q_1}{j} \left(\frac{2}{\tilde{t}'}\right)^j \left(-1 - \frac{t}{2}\right)_j \\
&\cdot \prod_{n=1} [i\sqrt{-t}(n-1)!]^{k_n} \prod_{m=2} \left[i\tilde{t}'(m-1)! \left(-\frac{1}{2M_2}\right) \right]^{q_m},
\end{aligned} \tag{11.24}$$

which is UV power-law behaved as expected. The summation in Eq.(11.24) can be represented by the Kummer function of the second kind U as follows,

$$\sum_{j=0}^p \binom{p}{j} \left(\frac{2}{\tilde{t}'}\right)^j \left(-1 - \frac{t}{2}\right)_j = 2^p (\tilde{t}')^{-p} U\left(-p, \frac{t}{2} + 2 - p, \frac{\tilde{t}'}{2}\right) \tag{11.25}$$

Finally, the amplitudes can be written as

$$\begin{aligned}
A^{(k_n, q_m)} &= \left(-\frac{i}{M_2}\right)^{q_1} U\left(-q_1, \frac{t}{2} + 2 - q_1, \frac{\tilde{t}'}{2}\right) B\left(-1 - \frac{s}{2}, -1 - \frac{t}{2}\right) \\
&\cdot \prod_{n=1} [i\sqrt{-t}(n-1)!]^{k_n} \prod_{m=2} \left[i\tilde{t}'(m-1)! \left(-\frac{1}{2M_2}\right) \right]^{q_m}.
\end{aligned} \tag{11.26}$$

In the above, U is the Kummer function of the second kind and is defined to be

$$U(a, c, x) = \frac{\pi}{\sin \pi c} \left[\frac{M(a, c, x)}{(a-c)!(c-1)!} - \frac{x^{1-c} M(a+1-c, 2-c, x)}{(a-1)!(1-c)!} \right] \quad (c \neq 2, 3, 4, \dots) \tag{11.27}$$

where $M(a, c, x) = \sum_{j=0}^{\infty} \frac{(a)_j x^j}{(c)_j j!}$ is the Kummer function of the first kind. U and M are the two solutions of the Kummer Equation

$$xy''(x) + (c-x)y'(x) - ay(x) = 0. \tag{11.28}$$

It is crucial to note that $c = \frac{t}{2} + 2 - q_1$, and is not a constant as in the usual case, so U in Eq.(11.26) is not a solution of the Kummer equation. This will make our analysis in the next section more complicated as we will see soon. On the contrary, since $a = -q_1$ an integer, the Kummer function in Eq.(11.25) terminated to be a finite sum. This will simplify the manipulation of Kummer function used in this chapter.

B. Reproducing ratios among hard scattering amplitudes

It can be seen from Eq.(11.24) that the Regge scattering amplitudes at each fixed mass level are no longer proportional to each other. The ratios are t dependent functions and can be calculated to be [63]

$$\begin{aligned} \frac{A^{(N,2m,q)}(s,t)}{A^{(N,0,0)}(s,t)} &= (-1)^m \left(-\frac{1}{2M_2}\right)^{2m+q} (\tilde{t}' - 2N)^{-m-q} (\tilde{t}')^{2m+q} \\ &\times \sum_{j=0}^{2m} (-2m)_j \left(-1 + N - \frac{\tilde{t}'}{2}\right)_j \frac{(-2/\tilde{t}')^j}{j!} + O\left\{\left(\frac{1}{\tilde{t}'}\right)^{m+1}\right\}, \end{aligned} \quad (11.29)$$

where $(x)_j = x(x+1)(x+2)\cdots(x+j-1)$ is the Pochhammer symbol which can be expressed in terms of the signed Stirling number of the first kind $s(n, k)$ as following

$$(x)_n = \sum_{k=0}^n (-1)^{n-k} s(n, k) x^k.$$

To ensure the identification for the general mass levels

$$\lim_{\tilde{t}' \rightarrow \infty} \frac{A^{(N,2m,q)}}{A^{(N,0,0)}} = \frac{T^{(N,2m,q)}}{T^{(N,0,0)}} = \left(-\frac{1}{M_2}\right)^{2m+q} \left(\frac{1}{2}\right)^{m+q} (2m-1)!!$$

suggested by the calculation for the mass level $M_2^2 = 4$, one needs the following identity

$$\begin{aligned} &\sum_{j=0}^{2m} (-2m)_j \left(-L - \frac{\tilde{t}'}{2}\right)_j \frac{(-2/\tilde{t}')^j}{j!} \\ &= 0 \cdot (-\tilde{t}')^0 + 0 \cdot (-\tilde{t}')^{-1} + \cdots + 0 \cdot (-\tilde{t}')^{-m+1} + \frac{(2m)!}{m!} (-\tilde{t}')^{-m} + O\left\{\left(\frac{1}{\tilde{t}'}\right)^{m+1}\right\} \end{aligned} \quad (11.30)$$

where $L = 1 - N$ and is an integer. Similar identification can be extended to the case of closed string as well [35]. For all four classes of high energy superstring scattering amplitudes, L is an integer too [64]. A recent work on string D-particle scattering amplitudes [42] also

gives an integer value of L . Note that L affects only the sub-leading terms in $O\left\{\left(\frac{1}{\tilde{t}'}\right)^{m+1}\right\}$. Here we give a simple example for $m = 3$ [64]

$$\begin{aligned} & \sum_{j=0}^6 (-2m)_j \left(-L - \frac{\tilde{t}'}{2}\right)_j \frac{(-2/\tilde{t}')^j}{j!} \\ &= \frac{120}{(-\tilde{t}')^3} + \frac{720L^2 - 2640L + 2080}{(-\tilde{t}')^4} + \frac{480L^4 - 4160L^3 + 12000L^2 - 12928L + 3840}{(-\tilde{t}')^5} \\ & \quad + \frac{64L^6 - 960L^5 + 5440L^4 - 14400L^3 + 17536L^2 - 7680L}{(-\tilde{t}')^6}. \end{aligned}$$

Mathematically, Eq.(11.30) was exactly proved [63, 64] for $L = 0, 1$ by a calculation based on a set of signed Stirling number identities developed recently in combinatorial theory in [144]. For general integer L cases, only the identity corresponding to the nontrivial leading term $\frac{(2m)!}{m!}(-\tilde{t}')^{-m}$ was rigorously proved in [64], but not for other “0 identities”. A numerical proof of Eq.(11.30) was given in [64] for arbitrary real values L and for non-negative integer m up to $m = 10$. It was then conjectured that [64] Eq.(11.30) is valid for any *real* number L and any non-negative integer m . It is important to prove Eq.(11.30) for any non-negative integer m and arbitrary real values L , since these values can be realized in the high energy scattering of *compactified* string states, as was shown recently in [67]. Real values of L appear in string compactifications due to the dependence on the generalized KK internal momenta K_i^{25} [67]

$$L = 1 - N - (K_2^{25})^2 + K_2^{25} K_3^{25}.$$

1. Proof of the new Stirling number identity

We now proceed to prove Eq.(11.30) [66]. We first rewrite the left-hand side of Eq.(11.30) in the following form

$$\begin{aligned} & \sum_{j=0}^{2m} (-2m)_j \left(-L - \frac{\tilde{t}'}{2}\right)_j \frac{(-2/\tilde{t}')^j}{j!} \\ &= \sum_{j=0}^{2m} (-1)^j \binom{2m}{j} \sum_{l=0}^j \binom{j}{l} (-L)_{j-l} \sum_{s=0}^l c(l, s) \left(-\frac{2}{\tilde{t}'}\right)^{j-s} \end{aligned} \quad (11.31)$$

where we have used the signless Stirling number of the first kind $c(l, s)$ to expand the Pochhammer symbol

$$(x)_n = \sum_{k=0}^n c(n, k) x^k \quad (11.32)$$

The coefficient of $(-2/\tilde{t}')^i$ in Eq.(11.31), which will be defined as $G(m, i)$, can be read off from the equation as

$$G(m, i) = \sum_{j=0}^{2m} \sum_{l=0}^j (-1)^{j+i} \binom{2m}{j} \binom{j}{l} (-L)_{j-l} c(l, j-i). \quad (11.33)$$

One needs to prove that

$$1. G(m, m) = (2m-1)!!, \text{ for all } L \in \mathbb{R}; \quad (11.34)$$

$$2. G(m, i) = 0, \text{ for all } L \in \mathbb{R} \text{ and } 0 \leq i < m. \quad (11.35)$$

From the definition of $c(n, k)$ in Eq.(11.32), we note that $c(n, k) \neq 0$ only if $0 \leq k \leq n$. Thus $c(l, j-i) \neq 0$ only if $j \geq i$ and $l \geq j-i$. We can rewrite $G(m, i)$ as

$$\begin{aligned} G(m, i) &= \sum_{j=i}^{2m} \sum_{l=j-i}^j (-1)^j \binom{2m}{j} \binom{j}{l} (-L)_{j-l} c(l, j-i) \\ &= \sum_{k=0}^{2m-i} \sum_{l=k}^{k+i} (-1)^{k+i} \binom{2m}{i+k} \binom{i+k}{l} (-L)_{k+i-l} c(l, k) \\ &= \sum_{k=0}^{2m-i} \sum_{p=0}^i (-1)^{k+i} \binom{2m}{i+k} \binom{i+k}{p+k} (-L)_{i-p} c(k+p, k) \\ &= \sum_{p=0}^i (-L)_{i-p} \sum_{k=0}^{2m-i} (-1)^{k+i} \binom{2m}{i+k} \binom{i+k}{p+k} c(k+p, k) \\ &= (-1)^i \sum_{p=0}^i (-L)_{i-p} \binom{2m}{i-p} \sum_{k=0}^{2m-i} (-1)^k \binom{2m-i+p}{k+p} c(k+p, k) \\ &\equiv (-1)^i \sum_{p=0}^i (-L)_{i-p} \binom{2m}{i-p} S_{2m-i}(p) \end{aligned} \quad (11.36)$$

where we have defined

$$S_N(p) = \sum_{k=0}^N (-1)^k \binom{N+p}{k+p} c(k+p, k). \quad (11.37)$$

It is easy to see that for fixed m and $0 \leq i < m$, $G(m, i)$ is a polynomial of L of degree i , expanded with the basis $1, (-L)_1, (-L)_2, \dots$. Note that $p \leq i < m$, so $2m-i \geq p+1$. For Eq.(11.35), we want to show that $S_N(p) = 0$ for $N \geq p+1$. For this purpose, we define the functions

$$C_n(x) = \sum_{k \geq 0} c(k+n, k) x^{k+n}. \quad (11.38)$$

The recurrence of the signless Stirling number identity

$$c(k+n, k) = (n+k-1)c(n+k-1, k) + c(n+k-1, k-1) \quad (11.39)$$

leads to the equation

$$C_n(x) = \frac{x^2}{1-x} \frac{d}{dx} C_{n-1}(x), \quad (11.40)$$

with the initial value

$$C_0(x) = \frac{1}{1-x}. \quad (11.41)$$

The first couple of $C_n(x)$ can be calculated to be

$$C_1(x) = \frac{x^2}{(1-x)^3}, \quad C_2(x) = \frac{x^4 + 2x^3}{(1-x)^5}, \quad C_3(x) = \frac{x^6 + 8x^5 + 6x^4}{(1-x)^7}. \quad (11.42)$$

Now by induction, it is easy to show that

$$C_n(x) = \frac{f_n(x)}{(1-x)^{2n+1}}, \quad \text{where } f_n(x) = x^{2n} + \mathcal{O}(x^{2n-1}), \quad (11.43)$$

and

$$f_n(1) = (2n-1)!!. \quad (11.44)$$

In order to prove Eq.(11.35), we note that $(-1)^N S_N(p)$ is the coefficient of x^{N+p} in the function

$$(1-x)^{N+p} C_p(x) = f_p(x) (1-x)^{N-p-1} = x^{N+p-1} + \mathcal{O}(\dots), \quad (11.45)$$

which is obviously zero for $N \geq p+1$. This proves $S_N(p) = 0$ for $N \geq p+1$ and thus Eq.(11.35).

In order to prove the first identity in Eq.(11.34), we first note that the above argument remains true for $i = m$ and $0 \leq p < i$. So Eq.(11.34) corresponds to the case $p = i = m$. By using Eq.(11.36), we can evaluate

$$G(m, m) = \sum_{k=0}^m (-1)^{k+m} \binom{2m}{k+m} \binom{k+m}{k+m} c(k+p, k) = \sum_{k=0}^m (-1)^{k+m} \binom{2m}{k+m} c(k+p, k). \quad (11.46)$$

Eq.(11.46) corresponds to the coefficient of x^{2m} in the function

$$(1-x)^{2m} C_m(x) = \frac{f_m(x)}{1-x} = f_m(x) (1+x+x^2+\dots). \quad (11.47)$$

By Eq.(11.44), this coefficient is

$$f_m(1) = (2m-1)!!. \quad (11.48)$$

This proves Eq.(11.34). We thus have completed the proof of Eq.(11.30) for any non-negative integer m and any real value L .

It was remarkable to first predict [63] the mathematical identities in Eq.(11.30) provided by string theory, and then a rigorous mathematical proof followed [66]. It was interesting to see that the validity of Eq.(11.30) includes non-integer values of L which were later realized by Regge string scatterings in compact space [67].

2. Subleading orders

In this section, we calculate the next few subleading order amplitudes in the RR for the mass level $M_2^2 = 4, 6$ [63]. We will see that the ratios in Eq.(5.16) and Eq.(5.40) persist to subleading order amplitudes in the RR. For the even mass levels with $(N-1) = \frac{M_2^2}{2} = \text{even}$, we conjecture and give evidences that the existence of these ratios in the RR persists to all orders in the Regge expansion of all high energy string scattering amplitudes. For the odd mass levels with $(N-1) = \frac{M_2^2}{2} = \text{odd}$, the existence of these ratios will show up only in the first $[N/2] + 1$ terms in the Regge expansion of the amplitudes.

We will extend the kinematic relations in the RR to the subleading orders. We first express all kinematic variables in terms of s and t , and then expand all relevant quantities in s :

$$E_1 = \frac{s - (m_2^2 + 2)}{2\sqrt{2}}, \quad (11.49)$$

$$E_2 = \frac{s + (m_2^2 + 2)}{2\sqrt{2}}, \quad (11.50)$$

$$|\mathbf{k}_2| = \sqrt{E_1^2 + 2}, \quad |\mathbf{K}_3| = \sqrt{\frac{s}{4} + 2}; \quad (11.51)$$

$$e_P \cdot k_1 = -\frac{1}{2m_2}s + \left(-\frac{1}{m_2} + \frac{m_2}{2}\right), \quad (\text{exact}) \quad (11.52)$$

$$\begin{aligned} e_L \cdot k_1 = & -\frac{1}{2m_2}s + \left(-\frac{1}{m_2} + \frac{m_2}{2}\right) - 2m_2s^{-1} - 2m_2(m_2^2 - 2)s^{-2} \\ & - 2m_2(m_2^4 - 6m_2^2 + 4)s^{-3} - 2m_2(m_2^6 - 12m_2^4 + 24m_2^2 - 8)s^{-4} + O(s^{-5}), \end{aligned} \quad (11.53)$$

$$e_T \cdot k_1 = 0. \quad (11.54)$$

A key step is to express the scattering angle θ in terms of s and t . This can be achieved by solving

$$t = - \left(- (E_2 - \frac{\sqrt{s}}{2})^2 + (|\mathbf{k}_2| - |\mathbf{k}_3| \cos \theta)^2 + |\mathbf{k}_3|^2 \sin^2 \theta \right) \quad (11.55)$$

to obtain

$$\theta = \arccos \left(\frac{s + 2t - m_2^2 + 6}{\sqrt{s + 8} \sqrt{\frac{(s+2)^2 - 2(s-2)m_2^2 + m_2^4}{s}}} \right). \text{ (exact)} \quad (11.56)$$

One can then calculate the following expansions

$$e_P \cdot k_3 = \frac{1}{m_2} (E_2 \frac{\sqrt{s}}{2} - |\mathbf{k}_2| |\mathbf{k}_3| \cos \theta) = - \frac{t + 2 - m_2^2}{2m_2}, \quad (11.57)$$

$$\begin{aligned} e_L \cdot k_3 &= \frac{1}{m_2} (k_2 \frac{\sqrt{2}}{2} - E_2 k_3 \cos \theta) \\ &= - \frac{t + 2 + m_2^2}{2m_2} - m_2 t s^{-1} - m_2 [-4(t + 1) + m_2^2(t - 2)] s^{-2} \\ &\quad - m_2 [4(4 + 3t) - 12t m_2^2 + (t - 4)m_2^4] s^{-3} \\ &\quad - m_2 [-16(3 + 2t) + 24(2 + 3t)m_2^2 \\ &\quad - 24(-1 + t)m_2^4 + (-6 + t)m_2^6] s^{-4} + O(s^{-5}), \end{aligned} \quad (11.58)$$

$$\begin{aligned} e_T \cdot k_3 &= -|\mathbf{k}_3| \sin \theta \\ &= -\sqrt{-t} - \frac{1}{2} \sqrt{-t} (2 + t + m_2^2) s^{-1} \\ &\quad - \frac{1}{8\sqrt{-t}} [32 + 52t + 20t^2 + t^3 + (32 + 20t - 6t^2)m_2^2 + (8 - 3t)m_2^4] s^{-2} \\ &\quad + \frac{1}{16\sqrt{-t}} [320 + 456t + 188t^2 + 22t^3 + t^4 - (-224 + 36t + 132t^2 + 5t^3)m_2^2 \\ &\quad + (-16 - 122t + 15t^2)m_2^4 + (-24 + 5t)m_2^6] s^{-3} \\ &\quad + \frac{1}{128(-t)^{3/2}} [1024 + 12032t + 16080t^2 + 7520t^3 + 1432t^4 + 136t^5 + 5t^6 \\ &\quad - 4(-512 - 896t + 2232t^2 + 1844t^3 + 170t^4 + 7t^5)m_2^2 \\ &\quad + 2(768 - 2240t - 2372t^2 + 1172t^3 + 35t^4)m_2^4 \\ &\quad - 4(-128 + 288t - 450t^2 + 35t^3)m_2^6 + (64 + 240t - 35t^2)m_2^8] s^{-4} + O(s^{-5}). \end{aligned} \quad (11.59)$$

We are now ready to calculate the expansions of the four amplitudes A_{TTT} , A_{LLT} , $A_{[LT]}$, $A_{(LT)}$ for the mass level $M_2^2 = 4$ to subleading orders in s in the RR. These are

$$A^{TTT} = \frac{1}{8} \sqrt{-t} t s^3 + \frac{3}{16} \sqrt{-t} t (t + 6) s^2 + \frac{3t^3 + 84t^2 - 68t - 864}{64} \sqrt{-t} s + O(1), \quad (11.60)$$

$$A^{LLT} = \frac{1}{64}\sqrt{-t}(t-6)s^3 + \frac{3}{128}\sqrt{-t}(t^2-20t-12)s^2 + \frac{3t^3-342t^2-92t+5016+1728(-t)^{-1/2}}{512}\sqrt{-t}s + O(1), \quad (11.61)$$

$$A^{[LT]} = -\frac{1}{64}\sqrt{-t}(t+2)s^3 - \frac{3}{128}\sqrt{-t}(t+2)^2s^2 - \frac{(3t-8)(t+6)^2[1-2(-t)^{-1/2}]}{512}\sqrt{-t}s + O(1), \quad (11.62)$$

$$A^{(LT)} = -\frac{1}{64}\sqrt{-t}(t+10)s^3 - \frac{1}{128}\sqrt{-t}(3t^2+52t+60)s^2 - \frac{3[t^3+30t^2+76t-1080-960(-t)^{-1/2}]}{512}\sqrt{-t}s + O(1). \quad (11.63)$$

One can now easily see that the ratios of the coefficients of the highest power of t in the leading order coefficient functions $\frac{1}{8} : \frac{1}{64} : -\frac{1}{64} : -\frac{1}{64}$ agree with the ratios in the GR $8 : 1 : -1 : -1$ calculated in Eq.(5.16) as expected. Moreover, one further observation is that these ratios remain the same for the coefficients of the highest power of t in the subleading orders (s^2) $\frac{3}{16} : \frac{3}{128} : -\frac{3}{128} : -\frac{3}{128}$ and (s) $\frac{3}{64} : \frac{3}{512} : -\frac{3}{512} : -\frac{3}{512}$. We conjecture that these ratios persist to all energy orders in the Regge expansion of the amplitudes. This is consistent with the results of GR by taking both $s, -t \rightarrow \infty$. For the mass level $M_2^2 = 6$ [28], the amplitudes can be calculated to be

$$A^{TTTT} = \frac{t^2}{16}s^4 + \frac{t^2(t+6)}{8}s^3 + \frac{t(t^3+24t^2-4t-256)}{16}s^2 + \frac{t(3t^3-2t^2-396t-768)}{4}s - \left(\frac{t^4}{4} + 166t^3 + 960t^2 - 64t - 1024\right)s^0 + (-83t^4 - 1536t^3 + 384t^2 + 21248t + 12288)s^{-1} + O(s^{-2}), \quad (11.64)$$

$$A^{TTLL} = \frac{t(t-16)}{192}s^4 + \frac{t(t^2-41t-32)}{96}s^3 + \frac{t^4-132t^3-328t^2+1984t+2048}{192}s^2 + \left(-\frac{11t^4}{32} - \frac{11t^3}{4} + \frac{163t^2}{3} + 184t + \frac{128}{3}\right)s^1 + \left(-\frac{11}{8}t^4 + 88t^3 + 744t^2 + 304t - 1408\right)s^0 + 4(11t^4 + 280t^3 + 204t^2 - 4448t - 4480)s^{-1} + O(s^{-2}), \quad (11.65)$$

$$\begin{aligned}
A^{LLLL} = & \frac{t(t-52)}{768}s^4 + \frac{t(t^2-140t+256)}{384}s^3 + \frac{t^4-456t^3+2816t^2-512t-16384}{768}s^2 \\
& \left(-\frac{19t^4}{64} + 6t^3 - \frac{17t^2}{3} - 176t - \frac{256}{3} \right) s^1 \\
& + (3t^4 - 10t^3 - 528t^2 - 672t + 1792)s^0 + O(s^{-1}), \tag{11.66}
\end{aligned}$$

$$\begin{aligned}
A^{TTL} = & -\frac{(t+20)t}{96\sqrt{6}}s^4 - \frac{t(t^2+31t+40)}{48\sqrt{6}}s^3 - \frac{t^4+38t^3+224t^2-1520t-2560}{96\sqrt{6}}s^2 \\
& + \frac{-3t^4-72t^3+2248t^2+12000t+5120}{48\sqrt{6}}s^1 \\
& + \frac{67t^3+1194t^2+1344t-3712}{2\sqrt{6}}s^0 + O(s^{-1}), \tag{11.67}
\end{aligned}$$

$$\begin{aligned}
A^{LLL} = & -\frac{t^2-8t-128}{384\sqrt{6}}s^4 - \frac{t^3-52t^2-412t+256}{192\sqrt{6}}s^3 \\
& - \frac{t^4-236t^3-1272t^2+4832t+15872}{384\sqrt{6}}s^2 \\
& + \frac{35t^4+50t^3-3008t^2-23728t-14848}{96\sqrt{6}}s^1 \\
& - \frac{47t^4+1432t^3+24796t^2+40640t-101376}{48\sqrt{6}}s^0 + O(s^{-1}), \tag{11.68}
\end{aligned}$$

$$\begin{aligned}
\tilde{A}^{LT,T} = & -\frac{t(t+2)}{64\sqrt{6}}s^4 - \frac{t(t+2)^2}{32\sqrt{6}}s^3 - \frac{t^4+12t^3+8t^2-152t-256}{64\sqrt{6}}s^2 \\
& + \frac{-3t^4+196t^2+624t+512}{32\sqrt{6}}s^1 + \sqrt{\frac{3}{8}}(5t^3+30t^2+24t-32)s^0 + O(s^{-1}), \tag{11.69}
\end{aligned}$$

$$\begin{aligned}
A^{LL} = & \frac{(t+8)^2}{384}s^4 + \frac{(t^3+20t^2+80t-128)}{192}s^3 + \frac{t^4+16t^3+96t^2-880t-3328}{384}s^2 \\
& + \frac{-t^4+8t^3-110t^2-1648t-1408}{48}s^1 \\
& + \frac{t^4-4t^3-202t^2-704t+1728}{6}s^0 + O(s^{-1}). \tag{11.70}
\end{aligned}$$

In the above calculations, as in the case of $M_2^2 = 4$, we have ignored a common overall factor which will be discussed in the next section. Note that the ratios of the coefficients in the leading order t for the energy orders s^4, s^3, s^2 reproduced the GR ratios in Eq.(5.16). However, the subleading terms for orders s^1, s^0 contain no GR ratios. Mathematically, this is because the highest power of t in the coefficient functions of s^1 is 4 rather than 5, and those

of s^0 is 4 rather than 6. This is because the power of t in the kinematic relation Eq.(11.59) can be as high as one wants if one goes to subleading orders, while that of Eq.(11.58) is not. The $\sin \theta$ factor in Eq.(11.59) contributes terms of higher order powers of t , while $\cos \theta$ factor in Eq.(11.58) does not. This can be seen from the kinematic relation in Eq.(11.56). In general, one can easily show that the $\sin \theta$ factor will contribute only for the even mass levels with $(N - 1) = \frac{M_2^2}{2} = \text{even}$.

We thus conjecture that the existence of the GR ratios in the RR persists to all orders in the Regge expansion of all string amplitudes for the even mass level. For the odd mass levels with $(N - 1) = \frac{M_2^2}{2} = \text{odd}$, the existence of the GR ratios will show up only in the first $[N/2] + 1$ terms in the Regge expansion of the amplitudes. An interesting question is whether this phenomena persists for the case of superstring where GSO projection needs to be imposed.

C. Universal power law behavior

In the discussion of the last section, we ignored an overall common factor $\frac{\Gamma(-1-s/2)\Gamma(-1-t/2)}{\Gamma(u/2+2)}$ of the amplitudes for mass levels $M_2^2 = 4, 6$. We paid attention only to the ratios among scattering amplitudes of different string states. In this section, we calculate the high energy behavior of string scattering amplitudes for string states at arbitrary mass levels in the RR. The power law behavior $\sim s^{\alpha(t)}$ of the four-tachyon amplitude in the RR is well known in the literature. Here we want to generalize this result to string states at arbitrary mass levels. We can use the saddle point method to calculate the leading term of gamma functions in the RR

$$\frac{\Gamma(-1-s/2)\Gamma(-1-t/2)}{\Gamma(u/2+2)} = \frac{\Gamma(-1-s/2)\Gamma(-1-t/2)}{\Gamma(-s/2-t/2+N-2)} \sim s^{t/2-N+1} \quad (\text{in the RR}). \quad (11.71)$$

Thus, the overall s -dependence in the amplitudes is of the form

$$A^{(k_n, q_m)} \sim s^{\alpha(t)} \quad (\text{in the RR}) \quad (11.72)$$

where

$$\alpha(t) = \alpha(0) + \alpha' t, \quad \alpha(0) = 1 \text{ and } \alpha' = 1/2. \quad (11.73)$$

This generalizes the high energy behavior of the four-tachyon amplitude in the RR to string states at arbitrary mass levels. The new result here is that the behavior is universal

and is mass level independent. In fact, as a simple application, one can also derive Eq.(11.72) directly from Eq.(11.26) by using

$$B\left(-1 - \frac{s}{2}, -1 - \frac{t}{2}\right) \sim s^{\alpha(t)}. \quad (\text{in the RR}) \quad (11.74)$$

We conclude that the well known $\sim s^{\alpha(t)}$ power-law behavior of the four tachyon string scattering amplitude in the RR can be extended to high energy string scattering amplitudes of arbitrary string states.

D. Recurrence relations of RSSA

To discuss relations among RSSA, one need to consider the complete RR string states [70]. The complete leading order high energy open string states in the Regge regime at each fixed mass level $N = \sum_{n,m,l>0} np_n + mq_m + lr_l$ are

$$|p_n, q_m, r_l\rangle = \prod_{n>0} (\alpha_{-n}^T)^{p_n} \prod_{m>0} (\alpha_{-m}^P)^{q_m} \prod_{l>0} (\alpha_{-l}^L)^{r_l} |0, k\rangle. \quad (11.75)$$

The case for $q_m = 0$ has been calculated previously in [63, 64] We stress that the inclusion of both α_{-m}^P and α_{-l}^L operators in Eq.(11.75) will be crucial to study Regge string Ward identities to be discussed in the later part of this chapter. It is also important to discuss the conformal invariant property of high energy string scattering amplitudes [64]. The momenta of the four particles on the scattering plane are

$$k_1 = \left(+\sqrt{p^2 + M_1^2}, -p, 0 \right), \quad (11.76)$$

$$k_2 = \left(+\sqrt{p^2 + M_2^2}, +p, 0 \right), \quad (11.77)$$

$$k_3 = \left(-\sqrt{q^2 + M_3^2}, -q \cos \phi, -q \sin \phi \right), \quad (11.78)$$

$$k_4 = \left(-\sqrt{q^2 + M_4^2}, +q \cos \phi, +q \sin \phi \right) \quad (11.79)$$

where $p \equiv |\tilde{p}|$, $q \equiv |\tilde{q}|$ and $k_i^2 = -M_i^2$. The relevant kinematics are

$$e^P \cdot k_1 \simeq -\frac{s}{2M_2}, \quad e^P \cdot k_3 \simeq -\frac{\tilde{t}}{2M_2} = -\frac{t - M_2^2 - M_3^2}{2M_2}; \quad (11.80)$$

$$e^L \cdot k_1 \simeq -\frac{s}{2M_2}, \quad e^L \cdot k_3 \simeq -\frac{\tilde{t}'}{2M_2} = -\frac{t + M_2^2 - M_3^2}{2M_2}; \quad (11.81)$$

and

$$e^T \cdot k_1 = 0, \quad e^T \cdot k_3 \simeq -\sqrt{-t} \quad (11.82)$$

where \tilde{t} and \tilde{t}' are related to t by finite mass square terms

$$\tilde{t} = t - M_2^2 - M_3^2, \quad \tilde{t}' = t + M_2^2 - M_3^2. \quad (11.83)$$

Note that, unlike the case of GR, here e^P does not approach to e^L in the RR. The Regge string scattering amplitudes can then be explicitly calculated to be

$$\begin{aligned} A(s, t) &\simeq \int_0^1 dy \, y^{k_1 k_2} (1-y)^{k_2 k_3} \cdot \prod_n \left[-\frac{(n-1)! e^T \cdot k_1}{(-y)^n} - \frac{(n-1)! e^T \cdot k_3}{(1-y)^n} \right]^{p_n} \\ &\quad \cdot \prod_m \left[\frac{(m-1)! e^P \cdot k_1}{(-y)^m} + \frac{(m-1)! e^P \cdot k_3}{(1-y)^m} \right]^{q_m} \\ &\quad \cdot \prod_l \left[-\frac{(l-1)! e^L \cdot k_1}{(-y)^l} - \frac{(l-1)! e^L \cdot k_3}{(1-y)^l} \right]^{r_l} \\ &\approx \int_0^1 dy \, y^{-\frac{s}{2}+N-2} (1-y)^{-\frac{t}{2}+N-2} \\ &\quad \cdot \prod_{n>0} \left[\frac{(n-1)! \sqrt{-t}}{(1-y)^n} \right]^{p_n} \cdot \prod_{m>1} \left[-\frac{(m-1)! \frac{\tilde{t}}{2M_2}}{(1-y)^m} \right]^{q_m} \cdot \prod_{l>1} \left[\frac{(l-1)! \frac{\tilde{t}'}{2M_2}}{(1-y)^l} \right]^{r_l} \\ &\quad \cdot \left[\frac{\frac{s}{2M_2}}{y} - \frac{\frac{\tilde{t}}{2M_2}}{(1-y)} \right]^{q_1} \left[-\frac{\frac{s}{2M_2}}{y} + \frac{\frac{\tilde{t}'}{2M_2}}{(1-y)} \right]^{r_1} \\ &= \prod_{n>0} [(n-1)! \sqrt{-t}]^{p_n} \cdot \prod_{m>1} \left[-(m-1)! \frac{\tilde{t}}{2M_2} \right]^{q_m} \cdot \prod_{l>1} \left[(l-1)! \frac{\tilde{t}'}{2M_2} \right]^{r_l} \\ &\quad \cdot \sum_{i,j} \binom{q_1}{i} \binom{r_1}{j} \left(-\frac{s}{\tilde{t}} \right)^i \left(-\frac{s}{\tilde{t}'} \right)^j B \left(-\frac{s}{2} + N - 1 - i - j, -\frac{t}{2} - 1 + i + j \right). \end{aligned} \quad (11.84)$$

In the second equality of the above equation, we have dropped the first term in the bracket with power of p_n , and the first terms in the brackets with powers of q_m and r_l for $m, l > 1$. These terms lead to subleading order terms in energy in the Regge limit [63, 64]. Now the beta function in Eq.(11.84) can be approximated in the RR by [63, 64]

$$B \left(-\frac{s}{2} + N - 1 - i - j, -\frac{t}{2} - 1 + i + j \right) = B \left(-\frac{s}{2} - 1, -\frac{t}{2} - 1 \right) \left(-\frac{s}{2} \right)^{-i-j} \left(-\frac{t}{2} - 1 \right)_{i+j} \quad (11.85)$$

where $(a)_j = a(a+1)(a+2)\dots(a+j-1)$ is the Pochhammer symbol. Finally we arrive at the amplitude with two equivalent expressions

$$A(s, t) = \prod_{n>0} [(n-1)! \sqrt{-t}]^{p_n} \cdot \prod_{m>0} \left[-(m-1)! \frac{\tilde{t}}{2M} \right]^{q_m} \cdot \prod_{l>1} \left[(l-1)! \frac{\tilde{t}'}{2M} \right]^{r_l} \quad (11.86)$$

$$\begin{aligned} & \cdot B\left(-\frac{s}{2} - 1, -\frac{t}{2} + 1\right) \left(\frac{1}{M}\right)^{r_1} \\ & \cdot \sum_{i=0}^{q_1} \binom{q_1}{i} \left(\frac{2}{\tilde{t}}\right)^i \left(-\frac{t}{2} - 1\right)_i U\left(-r_1, \frac{t}{2} + 2 - i - r_1, \frac{\tilde{t}'}{2}\right) \\ & = \prod_{n>0} [(n-1)! \sqrt{-t}]^{p_n} \cdot \prod_{m>1} \left[-(m-1)! \frac{\tilde{t}}{2M} \right]^{q_m} \cdot \prod_{l>0} \left[(l-1)! \frac{\tilde{t}'}{2M} \right]^{r_l} \quad (11.87) \\ & \cdot B\left(-\frac{s}{2} - 1, -\frac{t}{2} + 1\right) \left(-\frac{1}{M}\right)^{q_1} \\ & \cdot \sum_{j=0}^{r_1} \binom{r_1}{j} \left(\frac{2}{\tilde{t}'}\right)^j \left(-\frac{t}{2} - 1\right)_j U\left(-q_1, \frac{t}{2} + 2 - j - q_1, \frac{\tilde{t}}{2}\right). \end{aligned}$$

It is interesting to note that the Regge behavior is again universal and is mass level independent as in the case of previous section [63]

$$B\left(-1 - \frac{s}{2}, -1 - \frac{t}{2}\right) \sim s^{\alpha(t)} \quad (\text{in the RR}) \quad (11.88)$$

where $\alpha(t) = \alpha(0) + \alpha' t$, $\alpha(0) = 1$ and $\alpha' = 1/2$. That is, the well known $\sim s^{\alpha(t)}$ power-law behavior of the four tachyon string scattering amplitude in the RR can be extended to arbitrary higher string states. This result will be used to construct an inter-mass level recurrence relation for Regge string scattering amplitudes later in Eq.(13.56).

1. Recurrence relations and RR stringy Ward identities

In this section, we first discuss Regge stringy Ward identities derived from Regge string ZNS (RZNS) for mass level $M^2 = 2$ and 4 [70]. We will see that, unlike the case for the GR stringy Ward identities, Regge string Ward identities are not good enough to solve all Regge string scattering amplitudes algebraically. On the other hand, we found that the recurrence relations of Kummer functions Eq.(D.9) to Eq.(D.14) discussed in the appendix D can be used to prove all Regge stringy Ward identities. Presumably the calculation can be generalized to arbitrary mass levels. Another reason to work on recurrence relations of

Kummer functions instead of Regge stringy Ward identities is that the former is very easy to generalize to arbitrary higher mass levels while the latter is not.

Most importantly, for Kummer functions $U(a, c, x)$ in Regge string amplitudes in Eq.(11.86) and Eq.(11.87) with $a = -q_1$ (or $-r_1$) a non-positive integer, one can use recurrence relations to solve all $U(-q_1, c, x)$ functions algebraically and thus determine all Regge string scattering amplitudes at arbitrary mass levels algebraically up to multiplicative factors [70]. We stress that for general values of a , the best one can obtain from recurrence relations is to express any Kummer function in terms of any two of its associated function (see the appendix D).

There are 9 Regge string amplitudes for the mass level $M^2 = 2$, $A^{PP}(\alpha_{-1}^P \alpha_{-1}^P)$, $A^{PL}(\alpha_{-1}^P \alpha_{-1}^L)$, $A^{PT}(\alpha_{-1}^P \alpha_{-1}^T)$, $A^{LL}(\alpha_{-1}^L \alpha_{-1}^L)$, $A^{LT}(\alpha_{-1}^L \alpha_{-1}^T)$, $A^{TT}(\alpha_{-1}^T \alpha_{-1}^T)$, $A^P(\alpha_{-2}^P)$, $A^L(\alpha_{-2}^L)$, $A^T(\alpha_{-2}^T)$. For this mass level $\tilde{t} = t$, $\tilde{t}' = t + 4$. The Regge string ZNS (RZNS) in Eq.(E.6) and Eq.(E.7) gives two Regge stringy Ward identities

$$A^T - \sqrt{2}A^{PT} = 0, \quad (11.89)$$

$$A^L - \sqrt{2}A^{PL} = 0. \quad (11.90)$$

The RZNS in Eq.(E.5) gives

$$\sqrt{2}A^P - A^{PP} - \frac{1}{5}A^{LL} - \frac{1}{5}A^{TT} = 0. \quad (11.91)$$

It's obvious to see that these three Regge stringy Ward identities Eq.(E.6) to Eq.(E.5) are not good enough to solve all the 9 Regge string scattering amplitudes algebraically. Indeed, the amplitude A^{LT} does not even show up in any of these three Ward identities.

Instead of Regge stringy Ward identities, in the following we will do the calculation based on recurrence relations of Kummer functions. We want to prove these three Regge stringy Ward identities by using recurrence relations

$$U(a-1, c, x) - (2a-c+x)U(a, c, x) + a(1+a-c)U(a+1, c, x) = 0, \quad (11.92)$$

$$U(a, c, x) - aU(a+1, c, x) - U(a, c-1, x) = 0. \quad (11.93)$$

First, by taking some special values of arguments of Kummer function in Eq.(11.92) and Eq.(11.93), one easily obtain

$$U(-1, x, x) = 0, \quad (11.94)$$

$$U(-2, x, x) + xU(0, x, x) = 0 \quad (11.95)$$

and

$$U(0, c, x) - U(0, c-1, x) = 0. \quad (11.96)$$

By using Eq.(11.87), one easily see that the Ward identity Eq.(11.89) implies

$$U\left(0, \frac{t}{2} + 2, \frac{\tilde{t}}{2}\right) + U\left(-1, \frac{t}{2} + 1, \frac{\tilde{t}}{2}\right) = 0 \quad (11.97)$$

To prove Eq.(11.97) by recurrence relations, we note that for the case of $a = 0$, $c = \frac{t}{2} + 1$, $x = \frac{\tilde{t}}{2}$, Eq.(11.92) says

$$U\left(-1, \frac{t}{2} + 1, \frac{\tilde{t}}{2}\right) + U\left(0, \frac{t}{2} + 1, \frac{\tilde{t}}{2}\right) = 0. \quad (11.98)$$

We then apply Eq.(11.96) for the second term of Eq.(11.98) to obtain Eq.(11.97). This completes the proof of Regge stringy Ward identity Eq.(11.89) based on recurrence relations Eq.(11.92) and Eq.(11.93). The Ward identity in Eq.(11.90) implies

$$\frac{1}{\sqrt{2}} \frac{\tilde{t}'}{2} \left[U\left(0, \frac{t}{2} + 2, \frac{\tilde{t}}{2}\right) + U\left(-1, \frac{t}{2} + 1, \frac{\tilde{t}}{2}\right) \right] + \left(-\frac{t}{2} - 1\right) U\left(-1, \frac{t}{2}, \frac{\tilde{t}}{2}\right) = 0. \quad (11.99)$$

To prove Eq.(11.99) by using recurrence relations, we note that Eq.(11.97) implies the first and the second terms of Eq.(11.99) cancel out. Eq.(11.96) and $t = \tilde{t}$ say that the last term of Eq.(11.99) vanishes. Finally, to prove Eq.(11.91) by using recurrence relations, one needs to prove

$$\begin{aligned} & \left[\frac{1}{10} \left(\frac{\tilde{t}'}{2}\right)^2 + \frac{\tilde{t}}{2} - \frac{t}{5} \right] U\left(0, \frac{t}{2} + 2, \frac{\tilde{t}}{2}\right) + \frac{1}{2} U\left(-2, \frac{t}{2}, \frac{\tilde{t}}{2}\right) \\ & + \frac{1}{5} \left(\frac{\tilde{t}'}{2}\right) \left(-\frac{t}{2} - 1\right) U\left(0, \frac{t}{2} + 1, \frac{\tilde{t}}{2}\right) + \frac{1}{10} \left(-\frac{t}{2} - 1\right) \left(-\frac{t}{2}\right) U\left(0, \frac{t}{2}, \frac{\tilde{t}}{2}\right) = 0. \end{aligned} \quad (11.100)$$

Now Eq.(11.95) implies

$$U\left(0, \frac{t}{2} + 2, \frac{\tilde{t}}{2}\right) = U\left(0, \frac{t}{2} + 1, \frac{\tilde{t}}{2}\right) = U\left(0, \frac{t}{2}, \frac{\tilde{t}}{2}\right). \quad (11.101)$$

Therefore Eq.(11.100) is equivalent to

$$\frac{t}{2} U\left(0, \frac{t}{2}, \frac{\tilde{t}}{2}\right) + U\left(-2, \frac{t}{2}, \frac{\tilde{t}}{2}\right) = 0. \quad (11.102)$$

Finally one can use Eq.(11.95) and $\tilde{t} = t$ to prove Eq.(11.102). This completes the proof of Regge stringy Ward identities for mass level $M^2 = 2$ by using recurrence relations of Kummer functions.

We will give a brief description for the case of mass level $M^2 = 4$. There are 22 Regge string amplitudes for the mass level $M^2 = 4$, A^{PPP} , A^{PPL} , A^{PPT} , A^{PLL} , A^{PLT} , A^{PTT} , A^{PP} , A^{LP} , A^{TP} , A^{LLL} , A^{LLT} , A^{LTT} , A^{TTT} , A^{PL} , A^{LL} , A^{TL} , A^{PT} , A^{LT} , A^{TT} , A^P , A^L , A^T . To fix the notation, we adopt the convention of mass ordered in the α_{-n}^α operators, for example, $A^{LT}(\alpha_{-2}^L \alpha_{-1}^T)$ and $A^{TL}(\alpha_{-2}^T \alpha_{-1}^L)$ etc. For this mass level $\tilde{t} = t - 2$, $\tilde{t}' = t + 6$. The 8 RZNS Eqs.(E.12), (E.16), (E.17), (E.18), (E.20), (E.21), (E.22) and (E.23) calculated in the appendix E give 8 Regge stringy Ward identities

$$25A^{PPP} + 9A^{PLL} + 9A^{PTT} - 9A^{LL} - 9A^{TT} - 75A^{PP} + 50A^P = 0, \quad (11.103)$$

$$A^{PLL} - A^{LL} = 0, \quad (11.104)$$

$$A^{PTT} - A^{TT} = 0, \quad (11.105)$$

$$A^{PLT} - A^{(LT)} = 0, \quad (11.106)$$

$$9A^{PPT} + A^{LLT} + A^{TTT} - 18A^{(PT)} + 6A^T = 0, \quad (11.107)$$

$$9A^{PPL} + A^{LLL} + A^{LTT} - 18A^{(PL)} + 6A^L = 0, \quad (11.108)$$

$$A^{LLT} + A^{TTT} - 9A^{[PT]} - 3A^T = 0, \quad (11.109)$$

$$A^{LLL} + A^{LTT} - 9A^{[PL]} - 3A^L = 0. \quad (11.110)$$

It is obvious to see that these eight Regge stringy Ward identities are not good enough to solve the 22 Regge string scattering amplitudes algebraically. Indeed, for example, the amplitude $A^{[LT]}$ does not even show up in any of these eight Ward identities. However, in the GR, one can identify e^P and e^L components [26–28] (Correspondingly the creation operators α_{-n}^P and $-\alpha_{-n}^L$ are identified, where the sign comes from the difference between the timelike and spacelike directions specified by the metric of the scattering plane $\eta_{\mu\nu} = \text{diag}(-1, 1, 1)$.), and take high energy fixed angle limit to get three Ward identities in leading order energy

[26–28]

$$T^{LLT} + T^{(LT)} = 0, \quad (11.111)$$

$$10T^{LLT} + T^{TTT} + 18T^{(LT)} = 0, \quad (11.112)$$

$$T^{LLT} + T^{TTT} + 9T^{[LT]} = 0, \quad (11.113)$$

which can be easily solved to get [26–28]

$$T^{TTT} : T^{LLT} : T^{(LT)} : T^{[LT]} = 8 : 1 : -1 : -1. \quad (11.114)$$

The ratios above are consistent with Eq.(5.16).

For illustration, we now proceed to prove Regge stringy Ward identities Eq.(11.103) to Eq.(11.106) by using recurrence relations Eq.(11.92), Eq.(11.93) and

$$(c - a - 1)U(a, c - 1, x) - (x + c - 1)U(a, c, x) + xU(a, c + 1, x) = 0. \quad (11.115)$$

Other Regge stringy Ward identities Eq.(11.107) to Eq.(11.110) can be similarly proved by using recurrence relations. For the case of $a = -1$, $c = x + 1$, Eq.(11.115) reduces to

$$(x + 1)U(-1, x, x) - 2xU(-1, x + 1, x) + xU(-1, x + 2, x) = 0. \quad (11.116)$$

For the case of $a = -1$, $c = x + 2$, Eq.(11.93) reduces to

$$U(-1, x + 2, x) + U(0, x + 2, x) - U(-1, x + 1, x) = 0. \quad (11.117)$$

Finally Eq.(11.116), Eq.(11.117), and Eq.(11.94) say

$$U(-1, x + 2, x) = -2U(0, x + 2, x), \quad (11.118)$$

$$U(-1, x + 1, x) = -U(0, x + 2, x). \quad (11.119)$$

We are now ready to prove Regge stringy Ward identities. We first prove Regge stringy Ward identity Eq.(11.104). The two terms in Eq.(11.104) divided by the beta function can be calculated to be

$$\begin{aligned} \frac{1}{B}A^{PLL} &= -\frac{1}{M} \left(\frac{\tilde{t}}{2M} \right)^2 \left[U \left(-1, \frac{t}{2} + 1, \frac{t}{2} - 1 \right) + 2 \left(\frac{2}{\tilde{t}'} \right) \left(-\frac{t}{2} - 1 \right) U \left(-1, \frac{t}{2}, \frac{t}{2} - 1 \right) \right. \\ &\quad \left. + \left(\frac{2}{\tilde{t}'} \right)^2 \left(-\frac{t}{2} - 1 \right) \left(-\frac{t}{2} \right) U \left(-1, \frac{t}{2} - 1, \frac{t}{2} - 1 \right) \right] \\ &= -\frac{1}{M} \left(\frac{t+6}{2M} \right)^2 \left[U \left(-1, \frac{t}{2} + 1, \frac{t}{2} - 1 \right) - 2 \frac{t+2}{t+6} U \left(-1, \frac{t}{2}, \frac{t}{2} - 1 \right) \right. \\ &\quad \left. + \frac{t(t+2)}{(t+6)^2} U \left(-1, \frac{t}{2} - 1, \frac{t}{2} - 1 \right) \right], \end{aligned} \quad (11.120)$$

$$\begin{aligned}
\frac{1}{B}A^{LL} &= \left(\frac{\tilde{t}'}{2M}\right)^2 \left[U\left(0, \frac{t}{2} + 2, \frac{t}{2} - 1\right) + \left(\frac{2}{\tilde{t}'}\right) \left(-\frac{t}{2} - 1\right) U\left(0, \frac{t}{2} + 1, \frac{t}{2} - 1\right) \right] \\
&= \left(\frac{t+6}{2M}\right)^2 \left[U\left(0, \frac{t}{2} + 2, \frac{t}{2} - 1\right) - \frac{t+2}{t+6} U\left(0, \frac{t}{2} + 1, \frac{t}{2} - 1\right) \right] . \quad (11.121)
\end{aligned}$$

Therefore we want to show

$$\begin{aligned}
& -\frac{1}{M} \left[U\left(-1, \frac{t}{2} + 1, \frac{t}{2} - 1\right) - 2\frac{t+2}{t+6} U\left(-1, \frac{t}{2}, \frac{t}{2} - 1\right) + \frac{t(t+2)}{(t+6)^2} U\left(-1, \frac{t}{2} - 1, \frac{t}{2} - 1\right) \right] \\
& - U\left(0, \frac{t}{2} + 2, \frac{t}{2} - 1\right) + \frac{t+2}{t+6} U\left(0, \frac{t}{2} + 1, \frac{t}{2} - 1\right) \stackrel{?}{=} 0 \quad (11.122)
\end{aligned}$$

Eq.(11.94) implies the third term of Eq.(11.122) vanishes and therefore Eq.(11.96) implies that Eq.(11.122) is equivalent to

$$\begin{aligned}
& -\frac{1}{M} U\left(-1, \frac{t}{2} + 1, \frac{t}{2} - 1\right) + \frac{2}{M} \frac{t+2}{t+6} U\left(-1, \frac{t}{2}, \frac{t}{2} - 1\right) - \frac{4}{t+6} U\left(0, \frac{t}{2} + 1, \frac{t}{2} - 1\right) \\
& = \frac{1}{M} \left[-U\left(-1, \frac{t}{2} + 1, \frac{t}{2} - 1\right) + 2\frac{t+2}{t+6} U\left(-1, \frac{t}{2}, \frac{t}{2} - 1\right) - \frac{8}{t+6} U\left(0, \frac{t}{2} + 1, \frac{t}{2} - 1\right) \right] \\
& = 0 \quad (11.123)
\end{aligned}$$

For the case of $x = \frac{t}{2} - 1$, Eq.(11.118) and Eq.(11.119) implies

$$U\left(-1, \frac{t}{2} + 1, \frac{t}{2} - 1\right) = 2U\left(-1, \frac{t}{2}, \frac{t}{2} - 1\right) = -2U\left(0, \frac{t}{2} + 1, \frac{t}{2} - 1\right) . \quad (11.124)$$

Hence Eq.(11.123) is easily proved.

We now prove Regge stringy Ward identity Eq.(11.105). The two terms in Eq.(11.105) divided by the beta function can be calculated to be

$$\begin{aligned}
\frac{1}{B}A^{PTT} &= (-t) \left(-\frac{1}{M}\right) U\left(-1, \frac{t}{2} + 1, \frac{t}{2} - 1\right) , \\
\frac{1}{B}A^{TT} &= (-t) U\left(0, \frac{t}{2} + 2, \frac{t}{2} - 1\right) .
\end{aligned}$$

Therefore we want to show

$$\frac{t}{M} U\left(-1, \frac{t}{2} + 1, \frac{t}{2} - 1\right) + tU\left(0, \frac{t}{2} + 2, \frac{t}{2} - 1\right) \stackrel{?}{=} 0 \quad (11.125)$$

For the case of $x = \frac{t}{2} - 1$, Eq.(11.118) means

$$U\left(-1, \frac{t}{2} + 1, \frac{t}{2} - 1\right) = -2U\left(0, \frac{t}{2} + 1, \frac{t}{2} - 1\right) . \quad (11.126)$$

Eq.(11.126) and Eq.(11.96) prove Eq.(11.125).

We can now turn to prove Regge stringy Ward identity Eq.(11.103). We first note that Eq.(11.104) and Eq.(11.105) implies that Eq.(11.103) is equivalent to

$$25A^{PPP} - 75A^{PP} + 50A^P = 0. \quad (11.127)$$

The three terms in Eq.(11.127) divided by the beta function are

$$\frac{1}{B}A^P = \left(-\frac{t-2}{2M}\right) U\left(0, \frac{t}{2} + 2, \frac{t}{2} - 1\right), \quad (11.128)$$

$$\frac{1}{B}A^{PP} = -\frac{1}{M} \left(-\frac{t-2}{2M}\right) U\left(-1, \frac{t}{2} + 1, \frac{t}{2} - 1\right), \quad (11.129)$$

$$\frac{1}{B}A^{PPP} = \left(-\frac{1}{M}\right)^3 U\left(-3, \frac{t}{2} - 1, \frac{t}{2} - 1\right). \quad (11.130)$$

Therefore we want to show

$$\begin{aligned} & 2 \left(-\frac{t-2}{2M}\right) U\left(0, \frac{t}{2} + 2, \frac{t}{2} - 1\right) - 3 \left(-\frac{1}{M}\right) \left(-\frac{t-2}{2M}\right) U\left(-1, \frac{t}{2} + 1, \frac{t}{2} - 1\right) \\ & + \left(-\frac{1}{M}\right)^3 U\left(-3, \frac{t}{2} - 1, \frac{t}{2} - 1\right) \stackrel{?}{=} 0 \end{aligned} \quad (11.131)$$

For the case of $a = -2$, $c = x$, Eq.(11.92) gives

$$U(-3, x, x) + 4U(-2, x, x) + 2(1+x)U(-1, x, x) = 0. \quad (11.132)$$

Using Eq.(11.94), we obtain

$$U(-3, x, x) + 4U(-2, x, x) = 0. \quad (11.133)$$

From Eq.(11.133) and Eq.(11.95), we obtain

$$U(-3, x, x) - 4xU(0, x, x) = 0. \quad (11.134)$$

From Eq.(11.134) and Eq.(11.96), we obtain

$$\begin{aligned}
& 2 \left(-\frac{t-2}{2M} \right) U \left(0, \frac{t}{2} + 2, \frac{t}{2} - 1 \right) - 3 \left(-\frac{1}{M} \right) \left(-\frac{t-2}{2M} \right) U \left(-1, \frac{t}{2} + 1, \frac{t}{2} - 1 \right) \\
& + \left(-\frac{1}{M} \right)^3 U \left(-3, \frac{t}{2} - 1, \frac{t}{2} - 1 \right) \\
& = \left(2 \left(-\frac{t-2}{2M} \right) + 4 \left(\frac{t-2}{2} \right) \left(-\frac{1}{M} \right)^3 \right) U \left(0, \frac{t}{2} + 2, \frac{t}{2} - 1 \right) \\
& + 3 \frac{1}{M} \left(-\frac{t-2}{2M} \right) U \left(-1, \frac{t}{2} + 1, \frac{t}{2} - 1 \right) \\
& = \left(2 \left(-\frac{t-2}{4} \right) + \left(\frac{t-2}{2} \right) \left(-\frac{1}{2} \right) \right) U \left(0, \frac{t}{2} + 2, \frac{t}{2} - 1 \right) \\
& + \frac{3}{2} \left(-\frac{t-2}{4} \right) U \left(-1, \frac{t}{2} + 1, \frac{t}{2} - 1 \right) \\
& = \frac{t-2}{2} \left[-\frac{3}{2} U \left(0, \frac{t}{2} + 2, \frac{t}{2} - 1 \right) - \frac{3}{2} \frac{1}{2} U \left(-1, \frac{t}{2} + 1, \frac{t}{2} - 1 \right) \right] . \tag{11.135}
\end{aligned}$$

Finally Eq.(11.96) and Eq.(11.118) implies that Eq.(11.135) vanishes. This proves Eq.(11.131).

For the fourth stringy Ward identity at mass level $M^2 = 4$, the two terms in Eq.(11.106) divided by the beta function are

$$\begin{aligned}
\frac{1}{B} A^{PLT} &= -\frac{\sqrt{-t}\tilde{t}'}{2M^2} \left[U \left(-1, \frac{t}{2} + 1, \frac{\tilde{t}}{2} \right) + \left(\frac{2}{\tilde{t}'} \right) \left(-\frac{t}{2} - 1 \right) U \left(-1, \frac{t}{2}, \frac{\tilde{t}}{2} \right) \right] \\
&= -\frac{\sqrt{-t}(t+6)}{2M^2} \left[\begin{aligned} & U \left(-1, \frac{t}{2} + 1, \frac{t}{2} - 1 \right) \\ & + \left(\frac{2}{t+6} \right) \left(-\frac{t}{2} - 1 \right) U \left(-1, \frac{t}{2}, \frac{t}{2} - 1 \right) \end{aligned} \right] , \tag{11.136}
\end{aligned}$$

$$\frac{1}{B} A^{LT} = \sqrt{-t} \frac{\tilde{t}'}{2M} U \left(0, \frac{t}{2} + 2, \frac{\tilde{t}}{2} \right) = \sqrt{-t} \frac{t+6}{2M} U \left(0, \frac{t}{2} + 2, \frac{t}{2} - 1 \right) , \tag{11.137}$$

$$\begin{aligned}
\frac{1}{B} A^{TL} &= \sqrt{-t} \frac{\tilde{t}'}{2M} \left[U \left(0, \frac{t}{2} + 2, \frac{\tilde{t}}{2} \right) + \left(\frac{2}{\tilde{t}'} \right) \left(-\frac{t}{2} - 1 \right) U \left(0, \frac{t}{2} + 1, \frac{\tilde{t}}{2} \right) \right] \\
&= \sqrt{-t} \frac{t+6}{2M} \left[U \left(0, \frac{t}{2} + 2, \frac{t}{2} - 1 \right) + \left(\frac{2}{t+6} \right) \left(-\frac{t}{2} - 1 \right) U \left(0, \frac{t}{2} + 1, \frac{t}{2} - 1 \right) \right] . \tag{11.138}
\end{aligned}$$

Therefore we want to show

$$\begin{aligned}
& 2A^{PLT} - A^{LT} - A^{TL} \\
&= B\sqrt{t} \left[-\frac{t+6}{M^2} U \left(-1, \frac{t}{2} + 1, \frac{t}{2} - 1 \right) - \frac{2}{M^2} \left(-\frac{t}{2} - 1 \right) U \left(-1, \frac{t}{2}, \frac{t}{2} - 1 \right) \right. \\
& \quad \left. - \frac{t+6}{M} U \left(0, \frac{t}{2} + 2, \frac{t}{2} - 1 \right) - \frac{1}{M} \left(-\frac{t}{2} - 1 \right) U \left(0, \frac{t}{2} + 1, \frac{t}{2} - 1 \right) \right] \stackrel{?}{=} 0 . \tag{11.139}
\end{aligned}$$

Using Eq.(11.96), we obtain

$$\begin{aligned}
& \frac{1}{B}(2A^{PLT} - A^{LT} - A^{TL}) \\
&= \sqrt{t} \left[-\frac{t+6}{M^2} U\left(-1, \frac{t}{2} + 1, \frac{t}{2} - 1\right) - \frac{2}{M^2} \left(-\frac{t}{2} - 1\right) U\left(-1, \frac{t}{2}, \frac{t}{2} - 1\right) \right. \\
&\quad \left. + \frac{1}{M} \left(-t - 6 + \frac{t}{2} + 1\right) U\left(0, \frac{t}{2}, \frac{t}{2} - 1\right) \right] \\
&= \sqrt{t} \left[-\frac{t+6}{M^2} U\left(-1, \frac{t}{2} + 1, \frac{t}{2} - 1\right) - \frac{2}{M^2} \left(-\frac{t}{2} - 1\right) U\left(-1, \frac{t}{2}, \frac{t}{2} - 1\right) \right. \\
&\quad \left. + \frac{1}{M} \left(-\frac{t}{2} - 5\right) U\left(0, \frac{t}{2}, \frac{t}{2} - 1\right) \right] \\
&= \sqrt{t} \left[-\frac{t+6}{4} U\left(-1, \frac{t}{2} + 1, \frac{t}{2} - 1\right) + \frac{t+2}{4} U\left(-1, \frac{t}{2}, \frac{t}{2} - 1\right) \right. \\
&\quad \left. - \frac{t+10}{4} U\left(0, \frac{t}{2}, \frac{t}{2} - 1\right) \right] \tag{11.140}
\end{aligned}$$

One can now use Eq.(11.118) and Eq.(11.119) to prove that Eq.(11.140) vanishes. This completes the explicit proof of four Regge stringy Ward identities for mass level $M^2 = 4$ by using recurrence relations of Kummer functions. Other four Regge stringy Ward identities can be similarly proved.

2. Solving all RSSA by Kummer recurrence relations

We observe that the recurrence relations of Kummer functions are more powerful than Regge stringy Ward identities in relating Regge string scattering amplitudes. This is indeed the case as we will show [70] now in the following that all Regge string scattering amplitudes can be algebraically solved by using recurrence relations up to multiplicative factors in the first line of Eq.(11.86) (or Eq.(11.87)).

To be more precise, we will first show that the ratio

$$\frac{U(a, c, x)}{U(0, x, x)} = f(a, c, x), \quad a = 0, -1, -2, -3, \dots \tag{11.141}$$

is fixed and can be calculated by using recurrence relations Eq.(11.92), Eq.(11.93) and

$$(c - a)U(a, c, x) + U(a - 1, c, x) - xU(a, c + 1, x) = 0. \tag{11.142}$$

We stress that Eq.(11.141) is nontrivial in the sense that, for general values of a , the best one can obtain from recurrence relations is to express any Kummer function in terms of any two of its associated function (see Appendix D). However, Eq.(11.141) states that for non-positive integer values of a , $U(a, c, x)$ can be fixed up to an overall factor by using recurrence relations.

To prove Eq.(11.141), we first note that, for $a = 0, c = x$, recurrence relation Eq.(11.92) implies Eq.(11.94). This determines $\frac{U(a,x,x)}{U(0,x,x)}$ for a is a non-positive integer. For illustration, we list examples of relations

$$\begin{aligned} a = -1, U(-2, x, x) + 0 + xU(0, x, x) &= 0, \\ a = -2, U(-3, x, x) + 4U(-2, x, x) + 0 &= 0, \\ a = -3, U(-4, x, x) + 6U(-3, x, x) + 3(2+x)U(-2, x, x) &= 0, \\ &\dots\dots\dots \end{aligned}$$

which determines $\frac{U(-2,x,x)}{U(0,x,x)}, \frac{U(-3,x,x)}{U(0,x,x)}, \frac{U(-4,x,x)}{U(0,x,x)}, \dots$ recursively.

Next we extend the result to $\frac{U(a,c,x)}{U(0,x,x)}$ for $c = x + Z, Z = \text{integer}$. We first consider the simple case with $a = 0$. From Eq.(11.93), we obtain for $a = 0, c = x + i, i \in Z$

$$U(0, x + i, x) - U(0, x + i - 1, x) = 0, \quad (11.143)$$

which gives $\frac{U(0,x+i,x)}{U(0,x,x)} = 1$. This proves Eq.(11.141) for $a = 0$. For $a \in Z_-, c = x + Z_-$, we obtain from Eq.(11.93) with $c = x - i$

$$U(a, x - i, x) - aU(a + 1, x - i, x) - U(a, x - i - 1, x) = 0. \quad (11.144)$$

Since $\frac{U(a,x,x)}{U(0,x,x)}, \frac{U(a+1,x,x)}{U(0,x,x)}$ have been determined for $a \in Z_-$, this determines $\frac{U(a,x-i,x)}{U(0,x,x)}$ for $a \in Z_-, i = 1, 2, 3, \dots$ recursively. For $a \in Z_-, c = x + Z_+$, we obtain from Eq.(11.142) with $c = x + i$

$$(x - a + i)U(a, x + i, x) + U(a - 1, x + i, x) - xU(a, x + i + 1, x) = 0. \quad (11.145)$$

Since $\frac{U(a-1,x,x)}{U(0,x,x)}, \frac{U(a,x,x)}{U(0,x,x)}$ have been determined for $a \in Z_-$, this determines $\frac{U(a,x+i,x)}{U(0,x,x)}$ for $a \in Z_-, i = 1, 2, 3, \dots$ recursively. This completes the proof of Eq.(11.141) by using recurrence relations of Kummer functions.

Secondly, we want to show that each Kummer function in the summation of Eq.(11.87) can be expressed in terms of Regge string scattering amplitudes. To show this, we first consider $r_1 = 0$ amplitudes in a fixed mass level and a fixed q_1 with no summation over Kummer functions. These amplitudes contain only one Kummer function. Then let us take the amplitude with the maximum p_1 . By decreasing p_1 and increasing r_1 by 1, we can create an amplitude with two Kummer functions in the same mass level and the same q_1 . The

first one of the two Kummer functions is the one appeared in the previous amplitude with $r_1 = 0$, so we can write the second Kummer function in terms of the two amplitudes, one with $r_1 = 0$ and the other with $r_1 = 1$.

By decreasing p_1 and increasing r_1 by 1 again, we can create an amplitude with three Kummer functions in the same mass level and the same q_1 . The first two of the three Kummer functions is the ones appeared in the previous two amplitudes, so we can write the third Kummer functions in terms of the three amplitudes. We can repeat this process until $p_1 = 0$. In this way, we can express all the Kummer functions in Eq.(11.87) in terms of the RR amplitudes.

In the following, as an example, let us illustrate the above process for the mass level 4 amplitudes. There are 22 Regge string amplitudes for the mass level $M^2 = 4$. We first consider the group of amplitudes with $q_1 = 0$, $(T^{TTT}, T^{LTT}, T^{LLT}, T^{LLL})$. The corresponding r_1 for each amplitude are $(0, 1, 2, 3)$. By using Eq.(11.87), one can easily see that $U(0, \frac{t}{2} + 2, \frac{t}{2} - 1)$ can be expressed in terms of T^{TTT} , $U(0, \frac{t}{2} + 1, \frac{t}{2} - 1)$ can be expressed in terms of (T^{TTT}, T^{LTT}) , $U(0, \frac{t}{2}, \frac{t}{2} - 1)$ can be expressed in terms of $(T^{TTT}, T^{LTT}, T^{LLT})$, and finally $U(0, \frac{t}{2} - 1, \frac{t}{2} - 1)$ can be expressed in terms of $(T^{TTT}, T^{LTT}, T^{LLT}, T^{LLL})$.

Similarly, we can consider groups of amplitudes (T^{PT}, T^{PL}) , (T^{LT}, T^{LL}) and (T^{TT}, T^{TL}) with $q_1 = 0$; group of amplitude $(T^{PTT}, T^{PLT}, T^{PLL})$ with $q_1 = 1$ and group of amplitude (T^{PPT}, T^{PPL}) with $q_1 = 2$. All the remaining 7 amplitudes are with $r_1 = 0$, and each amplitude contains only one Kummer function. Due to the multiplicative factors, there are much more RR amplitudes than the number of Kummer functions involved at each fixed mass level. At mass level 4, for example, there are 22 RR amplitudes and only 10 Kummer functions involved. So there is an onto correspondence between RR amplitudes and Kummer functions. We have done the analysis by using Eq.(11.87). Similar analysis can be performed by using Eq.(11.86) to get the same results.

An important application of the above prescription is the construction of an infinite number of recurrence relations among Regge string scattering amplitudes. One can use the recurrence relations of Kummer functions Eq.(D.9) to Eq.(D.14) to systematically construct recurrence relations among Regge string scattering amplitudes.

Note that a simple calculation by using the explicit form of Kummer function in Eq.(15.12) gives $U(0, x, x) = 1$. However, when applying to the case of Regge string scattering amplitudes, it will bring back a multiplicative factor in the first line of Eq.(11.86),

(Eq.(11.87)) for each amplitude. We thus conclude that all Regge string scattering amplitudes can be algebraically solved by recurrence relations of Kummer functions up to multiplicative factors.

Finally we calculate some examples of recurrence relations among Regge string scattering amplitudes. At mass level $M^2 = 2$, by using Eq.(11.87) and the recurrence relation

$$U\left(-2, \frac{t}{2}, \frac{t}{2}\right) + \left(\frac{t}{2} + 1\right) U\left(-1, \frac{t}{2}, \frac{t}{2}\right) - \frac{t}{2} U\left(-1, \frac{t}{2} + 1, \frac{t}{2}\right) = 0, \quad (11.146)$$

one can obtain the following recurrence relation among Regge string scattering amplitudes [70]

$$M\sqrt{-t}A^{PP} - \frac{t}{2}A^{PT} = 0. \quad (11.147)$$

In contrast to the Regge stringy Ward identities Eq.(11.89), Eq.(11.90) and Eq.(11.91) which contain only constant coefficients, the recurrence relation in Eq.(11.147) contains kinematic variable t in its coefficients. Note that Eq.(11.147) is independent of all three Regge stringy Ward identities at mass level $M^2 = 2$.

At mass level $M^2 = 4$, by using Eq.(11.87), one can calculate

$$\frac{1}{B}A^{PPP} = \left(-\frac{1}{M}\right)^3 U\left(-3, \frac{t}{2} - 1, \frac{t}{2} - 1\right), \quad (11.148)$$

$$\frac{1}{B}A^{PPT} = \left(-\frac{1}{M}\right)^2 \sqrt{-t}U\left(-2, \frac{t}{2}, \frac{t}{2} - 1\right), \quad (11.149)$$

$$\frac{1}{B}A^{PPL} = \frac{t+6}{2M^3}U\left(-2, \frac{t}{2}, \frac{t}{2} - 1\right) + \frac{1}{M^3}\left(-\frac{t}{2} - 1\right)U\left(-2, \frac{t}{2} - 1, \frac{t}{2} - 1\right). \quad (11.150)$$

The recurrence relation

$$U\left(-3, \frac{t}{2} - 1, \frac{t}{2} - 1\right) + \left(\frac{t}{2} + 1\right)U\left(-2, \frac{t}{2} - 1, \frac{t}{2} - 1\right) - \left(\frac{t}{2} - 1\right)U\left(-2, \frac{t}{2}, \frac{t}{2} - 1\right) = 0 \quad (11.151)$$

leads to the following recurrence relation among Regge string scattering amplitudes [70]

$$M\sqrt{-t}A^{PPP} - 4A^{PPT} + M\sqrt{-t}A^{PPL} = 0. \quad (11.152)$$

We have explicitly verified Eq.(11.147) and Eq.(11.152). The generalization of Eq.(11.152) to arbitrary mass levels will be derived in Eq.(15.30) in chapter XV. It will be difficult to identify identity like Eq.(11.152) without using the recurrence relation Eq.(11.151). One can similarly construct infinite number of them for amplitudes at arbitrary higher mass levels

based on the recurrence relations of Kummer functions and their associated functions (see Appendix D).

For the third example, we construct an inter-mass level recurrence relation for Regge string scattering amplitudes at mass level $M^2 = 2, 4$. We begin with the addition theorem of Kummer function [71]

$$U(a, c, x + y) = \sum_{k=0}^{\infty} \frac{1}{k!} (a)_k (-1)^k y^k U(a + k, c + k, x) \quad (11.153)$$

which terminates to a finite sum for a non-positive integer a . By taking, for example, $a = -1, c = \frac{t}{2} + 1, x = \frac{t}{2} - 1$ and $y = 1$, the theorem gives

$$U\left(-1, \frac{t}{2} + 1, \frac{t}{2}\right) - U\left(-1, \frac{t}{2} + 1, \frac{t}{2} - 1\right) - U\left(0, \frac{t}{2} + 2, \frac{t}{2} - 1\right) = 0. \quad (11.154)$$

Note that, unlike all previous cases, the last arguments of Kummer functions in Eq.(11.154) can be different. Eq.(11.154) leads to an inter-mass level recurrence relation [70]

$$M(2)(t + 6)A_2^{TP} - 2M(4)^2\sqrt{-t}A_4^{LP} + 2M(4)A_4^{LT} = 0 \quad (11.155)$$

where masses $M(2) = \sqrt{2}, M(4) = \sqrt{4} = 2$, and A_2, A_4 are Regge string scattering amplitudes for mass levels $M^2 = 2, 4$ respectively. In deriving Eq.(11.155), it is important to use the fact that the Regge power law behavior in Eq.(10.27) is universal and is mass level independent [63]. Following the same procedure, one can construct infinite number of recurrence relations among Regge string scattering amplitudes at arbitrary mass levels which, in general, are independent of Regge stringy Ward identities.

In this chapter, we calculate the complete set of high energy string scattering amplitudes in the Regge regime. We derive Regge stringy Ward identities for the first few mass levels based on the decoupling of ZNS. These results are valid even for higher point functions and higher point loops as well by unitarity. We found that, unlike the case for the fixed angle regime, the Regge stringy Ward identities were not good enough to solve all the Regge string scattering amplitudes algebraically. On the other hand, we found that all the Regge stringy Ward identities can be explicitly proved by the recurrence relations of Kummer functions of the second kind. We then show that, instead of Regge stringy Ward identities, one can use these recurrence relations to solve all Regge string scattering amplitudes algebraically up to multiplicative factors.

Finally, for illustration, we calculate some examples of recurrence relations among Regge string scattering amplitudes of different string states based on recurrence relations and addition theorem of Kummer functions. In contrast to the Regge stringy Ward identities which contain only constant coefficients, these recurrence relations contains kinematic variable t in its coefficients and are in general independent of Regge stringy Ward identities. The dynamical origin of these recurrence relations remain to be studied. These recurrence relations among Regge string scattering amplitudes are dual to linear relations or symmetries among high energy fixed angle string scattering amplitudes discovered previously [26–28, 30, 31, 44].

Recently, five-point tachyon amplitude was considered in the context of BCFW application of string theory in [145]. It will be interesting to consider both RR and GR of higher spin five-point scattering amplitudes.

XII. FOUR CLASSES OF REGGE SUPERSTRING SCATTERING AMPLITUDES

In this chapter we will calculate [64] four classes of scattering amplitudes considered in chapter VIII corresponding to states in Eq.(8.1) to Eq.(8.4) in the RR. Moreover, we will extract from the RR superstring amplitudes the ratios among GR superstring amplitudes calculated in chapter VIII [33]

$$|N, 2m, q\rangle \otimes \left| b_{-\frac{3}{2}}^P \right\rangle = \left(-\frac{1}{2M_2} \right)^{q+m} \frac{(2m-1)!!}{(-M_2)^m} |N, 0, 0\rangle \otimes \left| b_{-\frac{3}{2}}^P \right\rangle, \quad (12.1)$$

$$|N+1, 2m+1, q\rangle \otimes \left| b_{-\frac{1}{2}}^P \right\rangle = \left(-\frac{1}{2M_2} \right)^{q+m} \frac{(2m+1)!!}{(-M_2)^{m+1}} |N, 0, 0\rangle \otimes \left| b_{-\frac{3}{2}}^P \right\rangle, \quad (12.2)$$

$$|N+1, 2m, q\rangle \otimes \left| b_{-\frac{1}{2}}^T \right\rangle = \left(-\frac{1}{2M_2} \right)^{q+m} \frac{(2m-1)!!}{(-M_2)^{m-1}} |N, 0, 0\rangle \otimes \left| b_{-\frac{3}{2}}^P \right\rangle, \quad (12.3)$$

$$|N-1, 2m, q-1\rangle \otimes \left| b_{-\frac{1}{2}}^T b_{-\frac{1}{2}}^P b_{-\frac{3}{2}}^P \right\rangle = \left(-\frac{1}{2M_2} \right)^{q+m} \frac{(2m-1)!!}{(-M_2)^m} |N, 0, 0\rangle \otimes \left| b_{-\frac{3}{2}}^P \right\rangle. \quad (12.4)$$

Note that, in order to simplify the notation, we have only shown the second state of the four point functions to represent the scattering amplitudes on both sides of each equation above. This notation will be used throughout the paper whenever is necessary. Eqs.(12.1) to (12.4) are thus the SUSY generalization of Eq.(5.60) for the bosonic string. There are much more high energy fermionic string scattering amplitudes other than states we will consider in this chapter.

We stress that, in addition to high energy scatterings of string states with polarizations orthogonal to the scattering plane considered previously in the GR [33], there are more high energy string scattering amplitudes with more worldsheet fermionic operators $b_{-\frac{n}{2}}^{P,T}$ in the string vertex.

A. Amplitude $|N, 2m, q\rangle \otimes |b_{-\frac{3}{2}}^P\rangle$

The first RR scattering amplitude we want to calculate corresponding to state in Eq.(8.3) is

$$A_1^{(N,2m,q)} = \langle \psi_1^{T^1} e^{-\phi_1} e^{ik_1 X_1} \cdot (\partial X_2^T)^{N-2m-2q} (\partial X_2^L)^{2m} (\partial^2 X_2^L)^q \partial \psi_2^P e^{-\phi_2} e^{ik_2 X_2} \cdot k_{\lambda 3} \psi_3^\lambda e^{ik_3 X_3} \cdot k_{\sigma 4} \psi_4^\sigma e^{ik_4 X_4} \rangle \quad (12.5)$$

where we have dropped out an overall factor. In Eq.(12.5), the first vertex is a vector state in the $(-)$ ghost picture, and the last two states are tachyons in the (0) ghost picture. The second state is a tensor in the $(-)$ ghost picture, so that the total superconformal ghost charges sum up to -2 . The $s - t$ channel of the amplitude can be calculated to be

$$A_1^{(N,2m,q)} = \int_0^1 dx x^{k_1 \cdot k_2} (1-x)^{k_2 \cdot k_3} \left[\frac{e^T \cdot k_3}{1-x} \right]^{N-2m-2q} \quad (12.6)$$

$$\cdot \left[\frac{e^P \cdot k_1}{-x} + \frac{e^P \cdot k_3}{1-x} \right]^{2m} \left[\frac{e^P \cdot k_1}{x^2} + \frac{e^P \cdot k_3}{(1-x)^2} \right]^q \cdot \frac{1}{x} \quad (12.7)$$

$$\cdot \left\{ \langle \psi_1^{T^1} \partial \psi_2^P \rangle \langle \psi_3^\lambda \psi_4^\sigma \rangle - \langle \psi_1^{T^1} \psi_3^\lambda \rangle \langle \partial \psi_2^P \psi_4^\sigma \rangle + \langle \psi_1^{T^1} \psi_4^\sigma \rangle \langle \partial \psi_2^P \psi_3^\lambda \rangle \right\} k_{\lambda 3} k_{\sigma 4} \quad (12.8)$$

$$\simeq \int_0^1 dx x^{k_1 \cdot k_2} (1-x)^{k_2 \cdot k_3} \left[\frac{e^T \cdot k_3}{1-x} \right]^{N-2m-2q} \quad (12.9)$$

$$\cdot \left[\frac{e^P \cdot k_1}{-x} + \frac{e^P \cdot k_3}{1-x} \right]^{2m} \left[\frac{e^P \cdot k_3}{(1-x)^2} \right]^q \cdot \frac{1}{x} \frac{1}{M_2} \left[-\frac{(e^T \cdot k_4)(k_2 \cdot k_3)}{(1-x)^2} \right]. \quad (12.10)$$

In Eq.(12.7), $\frac{e^P \cdot k_1}{x^2}$ is of subleading order in the RR and $\frac{1}{x}$ is the ghost contribution. The second term of Eq.(12.8) vanishes due to the $SL(2, R)$ gauge fixing $x_1 = 0, x_2 = x, x_3 = 1$ and $x_4 = \infty$. The first term of Eq.(12.8) vanishes due to $e^{T^1} \cdot e^{P^2} = 0$. The amplitude then

reduces to

$$\begin{aligned}
A_1^{(N,2m,q)} &\simeq \frac{\tilde{t}}{2M_2} (\sqrt{-t})^{N-2m-2q+1} \left(\frac{\tilde{t}}{2M_2} \right)^q \int_0^1 dx x^{k_1 \cdot k_2 - 1} (1-x)^{k_2 \cdot k_3 - N + 2m - 2} \\
&\cdot \sum_{j=0}^{2m} \binom{2m}{j} \left(\frac{s}{2M_2 x} \right)^j \left(\frac{-\tilde{t}}{2M_2(1-x)} \right)^{2m-j} \\
&= \frac{\tilde{t}}{2M_2} (\sqrt{-t})^{N-2m-2q+1} \left(\frac{\tilde{t}}{2M_2} \right)^{2m+q} \\
&\cdot \sum_{j=0}^{2m} \binom{2m}{j} (-1)^j \left(\frac{s}{\tilde{t}} \right)^j B(k_1 \cdot k_2 - j, k_2 \cdot k_3 - N + j - 1). \tag{12.11}
\end{aligned}$$

The Beta function above can be approximated in the large s , but fixed t limit as follows

$$\begin{aligned}
&B(k_1 \cdot k_2 - j, k_2 \cdot k_3 + j - N - 1) \\
&= B\left(1 - \frac{s}{2} + N - j, -\frac{1}{2} - \frac{t}{2} + j\right) \\
&= \frac{\Gamma(1 - \frac{s}{2} + N - j) \Gamma(-\frac{1}{2} - \frac{t}{2} + j)}{\Gamma(\frac{u}{2} - 1)} \\
&\approx B\left(1 - \frac{s}{2}, -\frac{1}{2} - \frac{t}{2}\right) \left(1 - \frac{s}{2}\right)^{N-j} \left(\frac{u}{2} - 1\right)^{-N} \left(-\frac{1}{2} - \frac{t}{2}\right)_j \\
&\approx B\left(1 - \frac{s}{2}, -\frac{1}{2} - \frac{t}{2}\right) \left(-\frac{s}{2}\right)^{-j} \left(-\frac{1}{2} - \frac{t}{2}\right)_j \tag{12.12}
\end{aligned}$$

where

$$(a)_j = a(a+1)(a+2)\dots(a+j-1) \tag{12.13}$$

is the Pochhammer symbol. The leading order amplitude in the RR can then be written as

$$\begin{aligned}
A_1^{(N,2m,q)} &\simeq \frac{\tilde{t}}{2M_2} B\left(1 - \frac{s}{2}, -\frac{1}{2} - \frac{t}{2}\right) \sqrt{-t}^{N-2m-2q+1} \left(\frac{1}{2M_2}\right)^{2m+q} \\
&\cdot (\tilde{t})^{2m+q} \sum_{j=0}^{2m} \binom{2m}{j} \left(\frac{2}{\tilde{t}}\right)^j \left(-\frac{1}{2} - \frac{t}{2}\right)_j, \tag{12.14}
\end{aligned}$$

which is UV power-law behaved as expected. The summation in Eq. (12.14) can be represented by the Kummer function of the second kind U as follows,

$$\sum_{j=0}^p \binom{p}{j} \left(\frac{2}{\tilde{t}}\right)^j \left(-\frac{1}{2} - \frac{t}{2}\right)_j = 2^p (\tilde{t})^{-p} U\left(-p, \frac{t}{2} - p + \frac{3}{2}, \frac{\tilde{t}}{2}\right). \tag{12.15}$$

Finally, the amplitudes can be written as

$$\begin{aligned}
A_1^{(N,2m,q)} &\simeq B\left(1 - \frac{s}{2}, -\frac{1}{2} - \frac{t}{2}\right) \sqrt{-t}^{N-2m-2q+1} \left(\frac{1}{2M_2}\right)^{2m+q+1} \\
&\cdot 2^{2m} (\tilde{t})^{q+1} U\left(-2m, \frac{t}{2} - 2m + \frac{3}{2}, \frac{\tilde{t}}{2}\right). \tag{12.16}
\end{aligned}$$

There are some important observations for the high energy amplitude in Eq.(12.16). First, the amplitude gives the universal power-law behavior for string states at *all* mass levels

$$A_1^{(N,2m,q)} \sim s^{\alpha(t)} \quad (\text{in the RR}) \quad (12.17)$$

where

$$\alpha(t) = a_0 + \alpha' t, \quad a_0 = \frac{1}{2} \text{ and } \alpha' = 1/2. \quad (12.18)$$

This generalizes the high energy behavior of the four massless vector amplitude in the RR to string states at arbitrary mass levels. Second, the amplitude gives the correct intercept $a_0 = \frac{1}{2}$ of fermionic string. Finally, the amplitude can be used to reproduce the ratios calculated in the GR as we will see in section E.

B. Amplitude $|N+1, 2m+1, q\rangle \otimes \left| b_{-\frac{1}{2}}^P \right\rangle$

Note that this is the only case with odd integer $2m+1$. The RR scattering amplitude corresponding to state in Eq.(8.2) can be written as

$$\begin{aligned} A_2^{(N+1,2m+1,q)} = & \langle \psi_1^{T1} e^{-\phi_1} e^{ik_1 X_1} \cdot (\partial X_2^T)^{N-2m-2q} (\partial X_2^L)^{2m+1} (\partial^2 X_2^L)^q \psi_2^P e^{-\phi_2} e^{ik_2 X_2} \\ & \cdot k_{\lambda 3} \psi_3^\lambda e^{ik_3 X_3} \cdot k_{\sigma 4} \psi_4^\sigma e^{ik_4 X_4} \rangle \end{aligned} \quad (12.19)$$

where we have dropped out an overall factor. The amplitude can be calculated to be

$$\begin{aligned}
A_2^{(N+1,2m+1,q)} &= \int_0^1 dx x^{k_1 \cdot k_2} (1-x)^{k_2 \cdot k_3} \left[\frac{e^T \cdot k_3}{1-x} \right]^{N-2m-2q} \\
&\cdot \left[\frac{e^P \cdot k_1}{-x} + \frac{e^P \cdot k_3}{1-x} \right]^{2m+1} \left[\frac{e^P \cdot k_1}{x^2} + \frac{e^P \cdot k_3}{(1-x)^2} \right]^q \cdot \frac{1}{x} \\
&\cdot \left\{ \langle \psi_1^{T^1} \psi_2^P \rangle \langle \psi_3^\lambda \psi_4^\sigma \rangle - \langle \psi_1^{T^1} \psi_3^\lambda \rangle \langle \psi_2^P \psi_4^\sigma \rangle + \langle \psi_1^{T^1} \psi_4^\sigma \rangle \langle \psi_2^P \psi_3^\lambda \rangle \right\} k_{\lambda 3} k_{\sigma 4} \\
&= \int_0^1 dx x^{k_1 \cdot k_2} (1-x)^{k_2 \cdot k_3} \left[\frac{e^T \cdot k_3}{1-x} \right]^{N-2m-2q} \left[\frac{e^P \cdot k_1}{-x} + \frac{e^P \cdot k_3}{1-x} \right]^{2m+1} \\
&\cdot \left[\frac{e^P \cdot k_3}{(1-x)^2} \right]^q \frac{1}{x} \frac{1}{M_2} \left[(e^{T^1} \cdot k_3)(k_2 \cdot k_4) - \frac{(e^{T^1} \cdot k_4)(k_2 \cdot k_3)}{1-x} \right] \\
&\simeq (-1)^N [\sqrt{-t}]^{N-2m-2q+1} \left(-\frac{1}{2M_2} \right)^{2m+q+2} \tilde{t}^{2m+q+1} \sum_{j=0}^{2m+1} \binom{2m+1}{j} \left(-\frac{s}{\tilde{t}} \right)^j \\
&\cdot \left[- (s+t+1) \int_0^1 dx x^{k_1 \cdot k_2 - j - 1} (1-x)^{k_2 \cdot k_3 - N + j - 1} \right. \\
&\quad \left. + \tilde{t} \int_0^1 dx x^{k_1 \cdot k_2 - j - 1} (1-x)^{k_2 \cdot k_3 - N + j - 2} \right] \\
&\simeq [\sqrt{-t}]^{N-2m-2q+1} \left(\frac{1}{2M_2} \right)^{2m+q+2} \tilde{t}^{2m+q+1} \sum_{j=0}^{2m+1} \binom{2m+1}{j} \left(-\frac{s}{\tilde{t}} \right)^j \\
&\cdot \left[- (s+t+1) B(k_1 \cdot k_2 - j, k_2 \cdot k_3 - N + j) \right. \\
&\quad \left. + \tilde{t} B(k_1 \cdot k_2 - j, k_2 \cdot k_3 - N + j - 1) \right]. \tag{12.20}
\end{aligned}$$

We then do an approximation for beta function similar to the calculation for $A_1^{(N,2m,q)}$ and end up with

$$\begin{aligned}
A_2^{(N+1,2m+1,q)} &\simeq B \left(1 - \frac{s}{2}, -\frac{1}{2} - \frac{t}{2} \right) [\sqrt{-t}]^{N-2m-2q+1} \left(\frac{1}{2M_2} \right)^{2m+q+2} \tilde{t}^{2m+q+1} \\
&\cdot \sum_{j=0}^{2m+1} \binom{2m+1}{j} \left[(1+t) \left(\frac{2}{\tilde{t}} \right)^j \left(\frac{1}{2} - \frac{t}{2} \right)_j - \tilde{t} \left(\frac{2}{\tilde{t}} \right)^j \left(-\frac{1}{2} - \frac{t}{2} \right)_j \right] \\
&\simeq B \left(1 - \frac{s}{2}, -\frac{1}{2} - \frac{t}{2} \right) [\sqrt{-t}]^{N-2m-2q+1} \left(\frac{1}{2M_2} \right)^{2m+q+2} 2^{2m+1} (\tilde{t})^q \\
&\cdot \left[(1+t) U \left(-1 - 2m, \frac{t}{2} - 2m - \frac{1}{2}, \frac{\tilde{t}}{2} \right) - \tilde{t} U \left(-1 - 2m, \frac{t}{2} - 2m + \frac{1}{2}, \frac{\tilde{t}}{2} \right) \right]. \tag{12.21}
\end{aligned}$$

Note that there are two terms in Eq.(12.21), and the first argument of the U function $a = -1 - 2m$ is odd. These differences will make the calculation of the ratios in the next section more complicated. Finally, the amplitude gives the universal power-law behavior for string states at *all* mass levels with the correct intercept $a_0 = \frac{1}{2}$ of fermionic string.

C. Amplitude $|N+1, 2m, q\rangle \otimes |b_{-\frac{1}{2}}^T\rangle$

The third RR scattering amplitude corresponding to state in Eq.(8.1) is

$$A_3^{(N+1, 2m, q)} = \langle \psi_1^{T1} e^{-\phi_1} e^{ik_1 X_1} \cdot (\partial X_2^T)^{N-2m-2q+1} (\partial X_2^L)^{2m} (\partial^2 X_2^L)^q \psi_2^T e^{-\phi_2} e^{ik_2 X_2} \cdot k_{\lambda 3} \psi_3^\lambda e^{ik_3 X_3} \cdot k_{\sigma 4} \psi_4^\sigma e^{ik_4 X_4} \rangle \quad (12.22)$$

where we have dropped out an overall factor. The scattering amplitude can be calculated to be

$$\begin{aligned} A_3^{(N+1, 2m, q)} &= \int_0^1 dx x^{k_1 \cdot k_2} (1-x)^{k_2 \cdot k_3} \left[\frac{e^T \cdot k_3}{1-x} \right]^{N-2m-2q+1} \\ &\quad \cdot \left[\frac{e^P \cdot k_1}{-x} + \frac{e^P \cdot k_3}{1-x} \right]^{2m} \left[\frac{e^P \cdot k_1}{x^2} + \frac{e^P \cdot k_3}{(1-x)^2} \right]^q \cdot \frac{1}{x} \\ &\quad \cdot \left\{ \langle \psi_1^{T1} \psi_2^T \rangle \langle \psi_3^\lambda \psi_4^\sigma \rangle - \langle \psi_1^{T1} \psi_3^\lambda \rangle \langle \psi_2^T \psi_4^\sigma \rangle + \langle \psi_1^{T1} \psi_4^\sigma \rangle \langle \psi_2^T \psi_3^\lambda \rangle \right\} k_{\lambda 3} k_{\sigma 4} \\ &= \int_0^1 dx x^{k_1 \cdot k_2} (1-x)^{k_2 \cdot k_3} \left[\frac{e^T \cdot k_3}{1-x} \right]^{N-2m-2q+1} \left[\frac{e^P \cdot k_1}{-x} + \frac{e^P \cdot k_3}{1-x} \right]^{2m} \left[\frac{e^P \cdot k_3}{(1-x)^2} \right]^q \\ &\quad \cdot \frac{1}{x} \left[\frac{(e^{T1} \cdot e^T)(k_3 \cdot k_4)}{-x} + (e^{T1} \cdot k_3)(e^T \cdot k_4) - \frac{(e^{T1} \cdot k_4)(e^T \cdot k_3)}{1-x} \right] \\ &\simeq [\sqrt{-t}]^{N-2m-2q+1} \left(\frac{1}{2M_2} \right)^{2m+q} \tilde{t}^{2m+q} \sum_{j=0}^{2m} \binom{2m}{j} \left(-\frac{s}{\tilde{t}} \right)^j \\ &\quad \cdot \left[-\frac{s}{2} \int_0^1 dx x^{k_1 \cdot k_2 - j - 2} (1-x)^{k_2 \cdot k_3 - N + j - 1} + t \int_0^1 dx x^{k_1 \cdot k_2 - j} (1-x)^{k_2 \cdot k_3 - N + j - 2} \right] \\ &\simeq [\sqrt{-t}]^{N-2m-2q+1} \left(\frac{1}{2M_2} \right)^{2m+q} \tilde{t}^{2m+q} \sum_{j=0}^{2m} \binom{2m}{j} \left(-\frac{s}{\tilde{t}} \right)^j \\ &\quad \cdot \left[\begin{aligned} &-\frac{s}{2} B(k_1 \cdot k_2 - j - 1, k_2 \cdot k_3 - N + j) \\ &+ t B(k_1 \cdot k_2 - j + 1, k_2 \cdot k_3 - N + j - 1) \end{aligned} \right]. \end{aligned} \quad (12.23)$$

We then do an approximation for beta function similar to the calculation for $A_1^{(N, 2m, q)}$ and end up with

$$\begin{aligned}
& \simeq -B \left(1 - \frac{s}{2}, -\frac{1}{2} - \frac{t}{2}\right) [\sqrt{-t}]^{N-2m-2q+1} \left(\frac{1}{2M_2}\right)^{2m+q} \tilde{t}^{2m+q} \\
& \cdot \sum_{j=0}^{2m} \binom{2m}{j} \left[\frac{(1+t)}{2} \left(\frac{2}{\tilde{t}}\right)^j \left(\frac{1}{2} - \frac{t}{2}\right)_j - t \left(\frac{2}{\tilde{t}}\right)^j \left(-\frac{1}{2} - \frac{t}{2}\right)_j \right] \\
& \simeq -B \left(1 - \frac{s}{2}, -\frac{1}{2} - \frac{t}{2}\right) [\sqrt{-t}]^{N-2m-2q+1} \left(\frac{1}{2M_2}\right)^{2m+q} 2^{2m-1} \tilde{t}^q \\
& \cdot \left[(1+t) U \left(-2m, \frac{t}{2} - 2m + \frac{1}{2}, \frac{\tilde{t}}{2}\right) - 2t U \left(-2m, \frac{t}{2} - 2m + \frac{3}{2}, \frac{\tilde{t}}{2}\right) \right]. \tag{12.24}
\end{aligned}$$

In this case there are again two terms as in the amplitude A_2 but with an even argument $a = -2m$. Finally, the amplitude gives the universal power-law behavior for string states at *all* mass levels with the correct intercept $a_0 = \frac{1}{2}$ of fermionic string.

D. Amplitude $|N-1, 2m, q-1\rangle \otimes |b_{-\frac{1}{2}}^T b_{-\frac{1}{2}}^P b_{-\frac{3}{2}}^P\rangle$

The fourth RR scattering amplitude corresponding to state in Eq.(8.4) is

$$\begin{aligned}
A_4^{(N-1, 2m, q-1)} &= \langle \psi_1^{T1} e^{-\phi_1} e^{ik_1 X_1} \cdot (\partial X_2^T)^{N-2m-2q} (\partial X_2^L)^{2m} (\partial^2 X_2^L)^{q-1} \psi_2^T \psi_2^P \partial \psi_2^P e^{-\phi_2} e^{ik_2 X_2} \\
&\quad \cdot k_{\lambda 3} \psi_3^\lambda e^{ik_3 X_3} \cdot k_{\sigma 4} \psi_4^\sigma e^{ik_4 X_4} \rangle \tag{12.25}
\end{aligned}$$

where we have dropped out an overall factor. The scattering amplitude can be calculated to be

$$\begin{aligned}
A_4^{(N-1,2m,q-1)} &= \int_0^1 dx x^{k_1 \cdot k_2} (1-x)^{k_2 \cdot k_3} \left[\frac{e^T \cdot k_3}{1-x} \right]^{N-2m-2q} \\
&\cdot \left[\frac{e^P \cdot k_1}{-x} + \frac{e^P \cdot k_3}{1-x} \right]^{2m} \left[\frac{e^P \cdot k_1}{x^2} + \frac{e^P \cdot k_3}{(1-x)^2} \right]^{q-1} \frac{1}{x} \\
&\cdot \langle \psi_1^{T^1} \psi_2^T \rangle \langle \psi_2^P \psi_4^\sigma \rangle \langle \partial \psi_2^P \psi_3^\lambda \rangle k_{\lambda 3} k_{\sigma 4} \\
&\simeq \int_0^1 dx x^{k_1 \cdot k_2} (1-x)^{k_2 \cdot k_3} \left[\frac{e^T \cdot k_3}{1-x} \right]^{N-2m-2q} \left[\frac{e^P \cdot k_1}{-x} + \frac{e^P \cdot k_3}{1-x} \right]^{2m} \\
&\cdot \left[\frac{e^P \cdot k_3}{(1-x)^2} \right]^{q-1} \frac{1}{x} \frac{1}{M_2^2} \left[\frac{(e^{T^1} \cdot e^T)(k_2 \cdot k_4)(k_2 \cdot k_3)}{(1-x)^2} \right] \\
&\simeq [\sqrt{-t}]^{N-2m-2q} \left(\frac{1}{2M_2} \right)^{2m+q+1} \tilde{t}^{2m+q} s \\
&\cdot \sum_{j=0}^{2m} \binom{2m}{j} \left(-\frac{s}{\tilde{t}} \right)^j \int_0^1 dx x^{k_1 \cdot k_2 - j} (1-x)^{k_2 \cdot k_3 - N + j - 1} \\
&\simeq [\sqrt{-t}]^{N-2m-2q} \left(\frac{1}{2M_2} \right)^{2m+q+1} \tilde{t}^{2m+q} s \\
&\cdot \sum_{j=0}^{2m} \binom{2m}{j} \left(-\frac{s}{\tilde{t}} \right)^j B(k_1 \cdot k_2 - j + 1, k_2 \cdot k_3 - N + j). \tag{12.26}
\end{aligned}$$

With a similar approximation for the beta function, we get

$$\begin{aligned}
A_4^{(N-1,2m,q-1)} &\simeq B \left(1 - \frac{s}{2}, -\frac{1}{2} - \frac{t}{2} \right) [\sqrt{-t}]^{N-2m-2q} \left(\frac{1}{2M_2} \right)^{2m+q+1} \tilde{t}^{2m+q} \\
&\cdot (1+t) \sum_{j=0}^{2m} \binom{2m}{j} \left(-\frac{2}{\tilde{t}} \right)^j \left(\frac{1}{2} - \frac{t}{2} \right)_j \\
&= B \left(1 - \frac{s}{2}, -\frac{1}{2} - \frac{t}{2} \right) [\sqrt{-t}]^{N-2m-2q} \left(\frac{1}{2M_2} \right)^{2m+q+1} \\
&\cdot 2^{2m} (\tilde{t})^q (1+t) U \left(-2m, \frac{t}{2} - 2m + \frac{1}{2}, \frac{\tilde{t}}{2} \right). \tag{12.27}
\end{aligned}$$

Again the amplitude gives the universal power-law behavior for string states at *all* mass levels with the correct intercept $a_0 = \frac{1}{2}$ of fermionic string. In the next section we are going to use the four amplitudes calculated in this section to extract ratios calculated in the fixed angle regime.

E. Reproducing ratios among hard SUSY scattering amplitudes

In the bosonic string calculation discussed in chapter XI[63], we learned that the relative coefficients of the highest power t terms in the leading order amplitudes in the RR can be used to reproduce the ratios of the amplitudes in the GR for each fixed mass level. In this section, we are going to generalize the calculation [64] to four classes of fermionic string states for arbitrary mass levels. We begin with the first amplitude of Eq.(12.16).

1. Ratios for $|N, 2m, q\rangle \otimes \left| b_{-\frac{3}{2}}^P \right\rangle$

It is important to note that there are no linear relations among high energy string scattering amplitudes, Eq.(12.16), of different string states for each fixed mass level in the RR. In other words, the ratios $A_1^{(N,2m,q)}/A_1^{(N,0,0)}$ are t -dependent functions and can be calculated to be

$$\begin{aligned} \frac{A_1^{(N,2m,q)}}{A_1^{(N,0,0)}} &= \left(-\frac{1}{2M_2} \right)^{2m+q} (-)^m (\tilde{t} + 2N + 1)^{-m-q} (\tilde{t})^{2m+q} \\ &\cdot \sum_{j=0}^{2m} (-2m)_j \left(-N - 1 - \frac{\tilde{t}}{2} \right)_j \frac{(-2/\tilde{t})^j}{j!} \end{aligned} \quad (12.28)$$

where we have used Eq.(11.3a) to replace t by \tilde{t} . If the leading order coefficients in Eq.(12.28) extracted from the amplitudes in the RR are to be identified with the ratios calculated in the GR in Eq.(12.1), we need the following identity

$$\sum_{j=0}^{2m} (-2m)_j \left(-L - \frac{\tilde{t}}{2} \right)_j \frac{(-2/\tilde{t})^j}{j!} \quad (12.29)$$

$$= 0(-\tilde{t})^0 + 0(-\tilde{t})^{-1} + \dots + 0(-\tilde{t})^{-m+1} + \frac{(2m)!}{m!} (-\tilde{t})^{-m} + O \left\{ \left(\frac{1}{\tilde{t}} \right)^{m+1} \right\} \quad (12.30)$$

where $L = N + 1$ and is an integer. This identity was proved in [66]. The coefficients of the terms $O \left\{ (1/\tilde{t})^{m+1} \right\}$ in Eq.(12.30) is irrelevant for string amplitudes. We thus have shown that high energy superstring scattering amplitudes $A_1^{(N,2m,q)}$ of Eq.(12.16) in the RR can be used to extract the ratios $T_1^{(N,2m,q)}/T_1^{(N,0,0)}$ of Eq.(12.1) in the GR by using the Stirling

number identities. That is

$$\begin{aligned} \lim_{t \rightarrow \infty} \frac{A_1^{(N,2m,q)}}{A_1^{(N,0,0)}} &= \lim_{t \rightarrow \infty} \left(-\frac{1}{2M_2} \right)^{2m+q} 2^{2m} (-t)^{m+2q} U \left(-2m, \frac{t}{2} - 2m + \frac{3}{2}, \frac{t}{2} \right) \\ &= \left(-\frac{1}{2M_2} \right)^{q+m} \frac{(2m-1)!!}{(-M_2)^m} = \frac{T_1^{(N,2m,q)}}{T_1^{(N,0,0)}}. \end{aligned} \quad (12.31)$$

2. Ratios for $|N+1, 2m+1, q\rangle \otimes |b_{-\frac{1}{2}}^P\rangle$

The ratios $A_2^{(N+1,2m+1,q)}/A_1^{(N,0,0)}$ can be calculated to be

$$\begin{aligned} \frac{A_2^{(N+1,2m+1,q)}}{A_1^{(N,0,0)}} &= \left(-\frac{1}{2M_2} \right)^{2m+q+1} (-\tilde{t})^m \cdot \left[(1+t) \sum_{j=0}^{2m+1} \binom{1+2m}{j} \left(\frac{2}{\tilde{t}} \right)^j \left(\frac{1}{2} - \frac{t}{2} \right)_j \right. \\ &\quad \left. - \tilde{t} \sum_{j=0}^{2m+1} \binom{1+2m}{j} \left(\frac{2}{\tilde{t}} \right)^j \left(-\frac{1}{2} - \frac{t}{2} \right)_j \right]. \end{aligned} \quad (12.32)$$

The bracket in the above equation can be simplified by dropping out the subleading order terms in the calculation, and one obtains

$$\begin{aligned} &(1+t) \sum_{j=0}^{2m+1} \binom{2m+1}{j} \left(\frac{2}{\tilde{t}} \right)^j \left(\frac{1}{2} - \frac{t}{2} \right)_j - \tilde{t} \sum_{j=0}^{2m+1} \binom{2m+1}{j} \left(\frac{2}{\tilde{t}} \right)^j \left(-\frac{1}{2} - \frac{t}{2} \right)_j \\ &= (1+t) \sum_{j=0}^{2m+1} (-2m-1)_j \left(-N - \frac{\tilde{t}}{2} \right)_j \frac{(-2/\tilde{t})^j}{j!} \\ &\quad - \tilde{t} \sum_{j=0}^{2m+1} (-2m-1)_j \left(-N - 1 - \frac{\tilde{t}}{2} \right)_j \frac{(-2/\tilde{t})^j}{j!} \\ &\approx \tilde{t} \sum_{j=0}^{2m+1} (-2m-1)_j \left(-N - \frac{\tilde{t}}{2} \right)_j \frac{(-2/\tilde{t})^j}{j!} - \tilde{t} \sum_{j=0}^{2m+1} (-2m-1)_j \left(-N - 1 - \frac{\tilde{t}}{2} \right)_j \frac{(-2/\tilde{t})^j}{j!} \\ &= 2(2m+1) \cdot \sum_{j=1}^{2m+1} (-2m)_{j-1} \left(-N - \frac{\tilde{t}}{2} \right)_{j-1} \frac{(-2/\tilde{t})^{j-1}}{(j-1)!} \\ &= 2(2m+1) \cdot \sum_{j=0}^{2m} (-2m)_j \left(-N - \frac{\tilde{t}}{2} \right)_j \frac{(-2/\tilde{t})^j}{j!} \end{aligned} \quad (12.33)$$

where we have dropped out the subleading order terms in the second equality of the calculation. Finally, the ratios can be calculated to be

$$\begin{aligned}
\frac{A_2^{(N+1,2m+1,q)}}{A_1^{(N,0,0)}} &= \left(-\frac{1}{2M_2}\right)^{2m+q+1} (-\tilde{t})^m \cdot \left[(1+t) \sum_{j=0}^{2m+1} \binom{1+2m}{j} \left(\frac{2}{\tilde{t}}\right)^j \left(\frac{1}{2} - \frac{t}{2}\right)_j \right. \\
&\quad \left. - \tilde{t} \sum_{j=0}^{2m+1} \binom{1+2m}{j} \left(\frac{2}{\tilde{t}}\right)^j \left(-\frac{1}{2} - \frac{t}{2}\right)_j \right] \\
&\simeq \left(-\frac{1}{2M_2}\right)^{2m+q+1} (-\tilde{t})^m 2(2m+1) \sum_{j=0}^{2m} (-2m)_j \left(-N-1-\frac{\tilde{t}}{2}\right)_j \frac{(-2/\tilde{t})^j}{j!}.
\end{aligned} \tag{12.34}$$

By using the identity Eq.(12.30), one can show that the leading order coefficients in Eq.(12.34) can be identified with the ratios calculated in the GR in Eq.(12.2). That is

$$\lim_{t \rightarrow \infty} \frac{A_2^{(N+1,2m+1,q)}}{A_1^{(N,0,0)}} = \frac{T_2^{(N+1,2m+1,q)}}{T_1^{(N,0,0)}}. \tag{12.35}$$

In the calculation for this case, it is crucial to reduce the upper limit of the summation $2m+1$ to $2m$ in Eq.(12.34). Otherwise, the identity Eq.(12.30) will not be applicable. It is remarkable to see that the leading order coefficients of Eq.(12.34) can be identified with ratios of Eq.(12.2) in the GR.

3. Ratios for $|N+1, 2m, q\rangle \otimes |b_{-\frac{1}{2}}^T\rangle$

The ratios $A_3^{(N+1,2m,q)}/A_1^{(N,0,0)}$ can be calculated to be

$$\frac{A_3^{(N+1,2m,q)}}{A_1^{(N,0,0)}} = \frac{1}{2} \left(-\frac{1}{2M_2}\right)^{2m+q-1} (-\tilde{t})^m \sum_{j=0}^{2m} (-2m)_j \left(-N-1-\frac{\tilde{t}}{2}\right)_j \frac{(-2/\tilde{t})^j}{j!}. \tag{12.36}$$

By using the identity Eq.(12.30), one can show that the leading order coefficients in Eq.(12.36) can be identified with the ratios calculated in the GR in Eq.(12.3). That is

$$\lim_{t \rightarrow \infty} \frac{A_3^{(N+1,2m,q)}}{A_1^{(N,0,0)}} = \frac{T_3^{(N+1,2m,q)}}{T_1^{(N,0,0)}}. \tag{12.37}$$

4. Ratios for $|N-1, 2m, q-1\rangle \otimes |b_{-\frac{1}{2}}^T b_{-\frac{1}{2}}^P b_{-\frac{3}{2}}^P\rangle$

The ratios $A_4^{(N-1,2m,q-1)}/A_1^{(N,0,0)}$ can be calculated to be

$$\frac{A_4^{(N-1,2m,q-1)}}{A_1^{(N,0,0)}} = \left(-\frac{1}{2M_2}\right)^{2m+q} (-\tilde{t})^m \sum_{j=0}^{2m} (-2m)_j \left(-N-1-\frac{\tilde{t}}{2}\right)_j \frac{(-2/\tilde{t})^j}{j!}. \tag{12.38}$$

By using the identity Eq.(12.30), one can show that the leading order coefficients in Eq.(12.38) can be identified with the ratios calculated in the GR in Eq.(12.4). That is

$$\lim_{t \rightarrow \infty} \frac{A_4^{(N+1, 2m+1, q)}}{A_1^{(N, 0, 0)}} = \frac{T_4^{(N+1, 2m+1, q)}}{T_1^{(N, 0, 0)}}. \quad (12.39)$$

We thus have succeeded in extracting the Ratios of high energy superstring scattering amplitudes in the GR from the high energy superstring scattering amplitudes in the RR. In the next section, we will study the subleading order amplitudes.

5. Subleading order amplitudes

In this section, we calculate the next few subleading order amplitudes in the RR for the mass levels $M_2^2 = 2(N+1) = 4, 6$. The calculation for $M_2^2 = 8$ can be found in [64]. The relevant kinematic can be found in the Appendix C. We will see that the ratios derived in the previous section persist to subleading order amplitudes in the RR. For the even mass levels with $(N+1) = \frac{M_2^2}{2} = \text{odd}$, we conjecture and give evidences that the existence of these ratios in the RR persists to all orders in the Regge expansion of all high energy string scattering amplitudes. For the odd mass levels with $(N+1) = \frac{M_2^2}{2} = \text{even}$, the existence of these ratios will show up only in the first $\frac{N+1}{2} + 1$ terms in the Regge expansion of the amplitudes. For the mass level $M_2^2 = 4$, there are three states for Eq.(8.1), and we obtain the subleading order expansions as follows.

$$\begin{aligned} |2, 0, 0\rangle |b_{-\frac{1}{2}}^T\rangle &\rightarrow \left(\frac{1}{4}t^2 - \frac{1}{4}t\right)s + \left(\frac{1}{4}t^3 + \frac{9}{4}t^2 + \frac{7}{4}t - \frac{5}{4}\right)s^0 \\ &\quad + \left(\frac{5}{2}t^3 + 18t^2 + \frac{39}{2}t + 4\right)s^{-1} + O[s^{-2}], \end{aligned} \quad (12.40)$$

$$\begin{aligned} |2, 2, 0\rangle |b_{-\frac{1}{2}}^T\rangle &\rightarrow \left(\frac{1}{32}t^2 + \frac{1}{8}t + \frac{19}{32}\right)s + \left(\frac{1}{32}t^3 + \frac{23}{32}t^2 + \frac{35}{32}t - \frac{19}{32}\right)s^0 \\ &\quad + \left(\frac{3}{4}t^3 - \frac{13}{4}t^2 - \frac{39}{4}t - \frac{23}{4}\right)s^{-1} + O[s^{-2}], \end{aligned} \quad (12.41)$$

$$\begin{aligned} |2, 0, 1\rangle |b_{-\frac{1}{2}}^T\rangle &\rightarrow \left(-\frac{1}{16}t^2 - \frac{1}{4}t + \frac{5}{16}\right)s + \left(-\frac{1}{16}t^3 - \frac{15}{16}t^2 - \frac{27}{16}t - \frac{29}{16}\right)s^0 \\ &\quad + \left(-\frac{3}{4}t^3 - \frac{17}{4}t^2 - \frac{45}{4}t - \frac{31}{4}\right)s^{-1} + O[s^{-2}]. \end{aligned} \quad (12.42)$$

In order to simplify the notation in the above equations, we have only shown the second state of the four-point functions in the correction functions to represent the scattering amplitudes

on the left hand side of each equation. We find that the ratios of the leading order coefficients of st^2 are $\frac{1}{4} : \frac{1}{32} : -\frac{1}{16}$, and it is easy to check that these are the same as the ratios in the fixed angle limit. Moreover, the ratios persist in the second subleading order terms $s^0 t^3$ as $\frac{1}{4} : \frac{1}{32} : -\frac{1}{16}$. The ratios terminate to this order. We can also compare the ratios among different worldsheet fermionic states but with the same mass level $M_2^2 = 4$. We have the expansions:

$$\begin{aligned} |2, 1, 0\rangle |b_{-\frac{1}{2}}^L\rangle &\rightarrow (\frac{1}{16}t^2 - \frac{7}{16}t)s + (\frac{1}{16}t^3 - \frac{29}{16}t^2 - \frac{49}{16}t - \frac{35}{16})s^0 \\ &\quad + (-\frac{7}{4}t^3 - \frac{67}{4}t^2 - \frac{117}{4}t - \frac{57}{4})s^{-1} + O[s^{-2}], \end{aligned} \quad (12.43)$$

$$\begin{aligned} |1, 0, 0\rangle |b_{-\frac{3}{2}}^L\rangle &\rightarrow (-\frac{1}{8}t^2 - \frac{5}{8}t)s + (-\frac{1}{8}t^3 - \frac{17}{8}t^2 - \frac{33}{8}t - \frac{25}{8})s^0 \\ &\quad + (-\frac{7}{4}t^3 - \frac{61}{4}t^2 - \frac{109}{4}t - \frac{55}{4})s^{-1} + O[s^{-2}], \end{aligned} \quad (12.44)$$

$$\begin{aligned} |0, 0, 0\rangle |b_{-\frac{1}{2}}^T b_{-\frac{1}{2}}^L b_{-\frac{3}{2}}^L\rangle &\rightarrow (\frac{1}{32}t^2 + \frac{3}{16}t + \frac{5}{32})s + (\frac{1}{32}t^3 + \frac{15}{32}t^2 + \frac{27}{32}t + \frac{13}{32})s^0 \\ &\quad + (\frac{1}{2}t^3 + \frac{7}{2}t^2 + \frac{11}{2}t + \frac{5}{2})s^{-1} + O[s^{-2}]. \end{aligned} \quad (12.45)$$

The ratios of the leading order coefficients are proportional to that of state $|2, 0, 0\rangle |b_{-\frac{1}{2}}^T\rangle$, and can be calculated to be

$$|2, 0, 0\rangle |b_{-\frac{1}{2}}^T\rangle : |2, 1, 0\rangle |b_{-\frac{1}{2}}^L\rangle : |1, 0, 0\rangle |b_{-\frac{3}{2}}^L\rangle : |0, 0, 0\rangle |b_{-\frac{1}{2}}^T b_{-\frac{1}{2}}^L b_{-\frac{3}{2}}^L\rangle = \frac{1}{4} : \frac{1}{16} : -\frac{1}{8} : \frac{1}{32}. \quad (12.46)$$

They again match with the ratios in the fixed angle limit. One can also find that the second subleading order ratios are the same $\frac{1}{4} : \frac{1}{16} : -\frac{1}{8} : \frac{1}{32}$. Again the ratios terminate to this order.

For the mass level $M_2^2 = 6$, there are three states in Eq.(8.1). We again calculate the subleading order expansions. Interestingly, in this case the ratios of the coefficients seem to be the same in all orders as can be seen in the following:

$$\begin{aligned} |3, 0, 0\rangle |b_{-\frac{1}{2}}^T\rangle &\rightarrow \sqrt{-t}(\frac{1}{8}t^2 - \frac{1}{8}t)s^2 + \sqrt{-t}(\frac{3}{16}t^3 + \frac{25}{16}t^2 + \frac{25}{16}t - \frac{21}{16})s \\ &\quad + \sqrt{-t}(\frac{3}{64}t^4 + \frac{197}{64}t^3 + \frac{625}{32}t^2 + \frac{743}{32}t + \frac{411}{64})s^0 + O[s^{-1}], \end{aligned} \quad (12.47)$$

$$\begin{aligned} |3, 2, 0\rangle |b_{-\frac{1}{2}}^T\rangle &\rightarrow \sqrt{-t}(\frac{1}{96}t^2 - \frac{1}{48}t + \frac{11}{32})s^2 + \sqrt{-t}(\frac{1}{64}t^3 + \frac{13}{32}t - \frac{5}{8})s \\ &\quad + \sqrt{-t}(\frac{1}{256}t^4 + \frac{9}{128}t^3 - \frac{925}{256}t^2 - \frac{729}{64}t - \frac{1481}{256})s^0 + O[s^{-1}], \end{aligned} \quad (12.48)$$

$$\begin{aligned}
|3, 0, 1\rangle|b_{-\frac{1}{2}}^T\rangle &\rightarrow \sqrt{-t}\left(-\frac{1}{16\sqrt{6}}t^2 - \frac{3}{8\sqrt{6}}t + \frac{7}{16\sqrt{6}}\right)s^2 \\
&+ \sqrt{-t}\left(-\frac{3}{32\sqrt{6}}t^3 - \frac{3}{2\sqrt{6}}t^2 - \frac{51}{16\sqrt{6}}t - \frac{19}{4\sqrt{6}}\right)s \\
&+ \sqrt{-t}\left(-\frac{3}{128\sqrt{6}}t^4 - \frac{111}{64\sqrt{6}}t^3 - \frac{1841}{128\sqrt{6}}t^2 - \frac{1209}{32\sqrt{6}}t - \frac{3573}{128\sqrt{6}}\right)s^0 + O[s^{-1}].
\end{aligned} \tag{12.49}$$

We find that the ratios of the leading order coefficients of $s^2t^{5/2}$ are $\frac{1}{8} : \frac{1}{96} : -\frac{1}{16\sqrt{6}}$, and they agree with the ratios in the fixed angle limit. The ratios of the second and the third order coefficients of $st^{7/2}$ and $s^0t^{9/2}$ are $\frac{3}{16} : \frac{1}{64} : -\frac{3}{32\sqrt{6}}$ and $\frac{3}{64} : \frac{1}{256} : -\frac{3}{128\sqrt{6}}$, respectively. We find that these two set of ratios are the same with one another. We predict that the ratios persist to all orders in the expansions.

The expansions among different worldsheet fermionic states but with same mass level $M_2^2 = 6$ are

$$\begin{aligned}
|3, 0, 0\rangle|b_{-\frac{1}{2}}^L\rangle &\rightarrow \sqrt{-t}\left(\frac{1}{48}t^2 - \frac{17}{48}t\right)s^2 + \sqrt{-t}\left(\frac{1}{32}t^3 - \frac{151}{96}t^2 - \frac{295}{96}t - \frac{119}{32}\right)s \\
&+ \sqrt{-t}\left(\frac{1}{128}t^4 - \frac{249}{128}t^3 - \frac{1317}{64}t^2 - \frac{2883}{64}t - \frac{3831}{128}\right)s^0 + O[s^{-1}], \tag{12.50} \\
|2, 0, 0\rangle|b_{-\frac{3}{2}}^L\rangle &\rightarrow \sqrt{-t}\left(-\frac{1}{8\sqrt{6}}t^2 - \frac{7}{8\sqrt{6}}t\right)s^2 \\
&+ \sqrt{-t}\left(-\frac{3}{16\sqrt{6}}t^3 - \frac{57}{16\sqrt{6}}t^2 - \frac{129}{16\sqrt{6}}t - \frac{147}{16\sqrt{6}}\right)s \\
&+ \sqrt{-t}\left(-\frac{3}{64\sqrt{6}}t^4 - \frac{285}{64\sqrt{6}}t^3 - \frac{1289}{32\sqrt{6}}t^2 - \frac{2831}{32\sqrt{6}}t - \frac{4011}{64\sqrt{6}}\right)s^0 + O[s^{-1}],
\end{aligned} \tag{12.51}$$

$$\begin{aligned}
|1, 0, 0\rangle|b_{-\frac{1}{2}}^T b_{-\frac{1}{2}}^L b_{-\frac{3}{2}}^L\rangle &\rightarrow \sqrt{-t}\left(\frac{1}{96}t^2 + \frac{1}{12}t + \frac{7}{96}\right)s^2 + \sqrt{-t}\left(\frac{1}{64}t^3 + \frac{9}{32}t^2 + \frac{31}{48}t + \frac{61}{96}\right)s \\
&+ \sqrt{-t}\left(\frac{1}{256}t^4 + \frac{77}{192}t^3 + \frac{2531}{768}t^2 + \frac{643}{96}t + \frac{3569}{768}\right)s^0 + O[s^{-1}]. \tag{12.52}
\end{aligned}$$

The ratios of the leading order coefficients are given by

$$\begin{aligned}
|3, 0, 0\rangle|b_{-\frac{1}{2}}^T\rangle : |3, 1, 0\rangle|b_{-\frac{1}{2}}^L\rangle : |2, 0, 0\rangle|b_{-\frac{3}{2}}^L\rangle : |1, 0, 0\rangle|b_{-\frac{1}{2}}^T b_{-\frac{1}{2}}^L b_{-\frac{3}{2}}^L\rangle \\
= \frac{1}{8} : \frac{1}{48} : -\frac{1}{8\sqrt{6}} : \frac{1}{96}.
\end{aligned} \tag{12.53}$$

We have checked that they agree with the ratios in the fixed angle limit. The second and the third subleading order ratios are $\frac{3}{16} : \frac{1}{32} : -\frac{3}{16\sqrt{6}} : \frac{1}{64}$ and $\frac{3}{64} : \frac{1}{128} : -\frac{3}{64\sqrt{6}} : \frac{1}{256}$, respectively. Again they agree with the ratios in the fixed angle limit. We expect that the ratios persist to all orders in the expansions.

XIII. RECURRENCE RELATIONS OF HIGHER SPIN BPST VERTEX OPERATORS

In this chapter, we study higher spin Regge string scattering amplitudes from BPST vertex operator approach [72]. Note that in the original BPST paper [61], the authors calculated the case of four tachyon closed string and thus Pomeron vertex operators. Here, for simplicity, we will calculate higher spin BPST vertex operators at arbitrary mass levels of open bosonic string. The calculation can be easily generalized to closed string case.

We find that all BPST vertex operators can be expressed in terms of Kummer functions of the second kind. We can then derive infinite number of recurrence relations among BPST vertex operators of different string states. These recurrence relations among BPST vertex operators lead to the recurrence relations among Regge string scattering amplitudes discovered in chapters XI and XV. [70, 73].

A. Four tachyon scattering

We will calculate high energy open string scatterings in the Regge Regime

$$s \rightarrow \infty, \sqrt{-t} = \text{fixed (but } \sqrt{-t} \neq \infty) \quad (13.1)$$

where

$$s = -(k_1 + k_2)^2 \text{ and } t = -(k_2 + k_3)^2. \quad (13.2)$$

Note that the convention for s and t adopted here is different from the original BPST paper in [61].

We first review the calculation of tachyon BPST vertex operator [61]. The $s - t$ channel of open string four tachyon amplitude can be written as

$$A = \int_0^1 d\omega \cdot \omega^{k_1 \cdot k_2} (1 - \omega)^{k_2 \cdot k_3} = \int_0^1 d\omega \cdot \omega^{-2 - \frac{s}{2}} (1 - \omega)^{-2 - \frac{t}{2}}. \quad (13.3)$$

Since $s \rightarrow \infty$, the integral is dominated around $\omega = 1$. Making the variable transformation $\omega = 1 - x$, the integral is dominated around $x = 0$, we obtain

$$A = \int_0^1 dx \cdot (1 - x)^{-2 - \frac{s}{2}} x^{-2 - \frac{t}{2}} \simeq \int dx \cdot x^{-2 - \frac{t}{2}} e^{\frac{s}{2}x} = \Gamma\left(-1 - \frac{t}{2}\right) \left(-\frac{s}{2}\right)^{1 + \frac{t}{2}}. \quad (13.4)$$

Alternatively, the integral in A can be expressed as

$$A = \int d\omega \langle e^{ik_1 X(0)} e^{ik_2 X(\omega)} e^{ik_3 X(1)} e^{ik_4 X(\infty)} \rangle. \quad (13.5)$$

One can calculate the operator product expansion (OPE) in the Regge limit

$$e^{ik_2 X(w)} e^{ik_3 X(z)} \sim |w - z|^{k_2 \cdot k_3} e^{i(k_2 + k_3)X(z) + ik_2(w-z)\partial X(z) + \dots}.$$

This means

$$e^{ik_2 X(w)} e^{ik_3 X(1)} \sim (1 - \omega)^{k_2 \cdot k_3} e^{ikX(1) - ik_2(1-\omega)\partial X(1) + \text{higher power of } (1-\omega)}, k = k_2 + k_3 \quad (13.6)$$

In evaluating Eq.(13.5), one can instead carry out the ω integration first in Eq.(13.6) at the operator level to obtain the BPST vertex operator [61]

$$\begin{aligned} V_{BPST} &= \int d\omega e^{ik_2 X(\omega)} e^{ik_3 X(1)} \\ &\sim \int d\omega (1 - \omega)^{k_2 \cdot k_3} e^{ikX(1) - ik_2(1-\omega)\partial X(1)} \\ &= \int dx x^{k_2 \cdot k_3} e^{ikX(1) - ik_2 x \partial X(1)} \\ &= \Gamma\left(-1 - \frac{t}{2}\right) [ik_2 \partial X(1)]^{1 + \frac{t}{2}} e^{ikX(1)}, \end{aligned} \quad (13.7)$$

which leads to the same amplitude as in Eq.(13.4)

$$\begin{aligned} A &= \langle e^{ik_1 X(0)} V_P e^{ik_4 X(\infty)} \rangle \\ &= \Gamma\left(-1 - \frac{t}{2}\right) \langle e^{ik_1 X(0)} [ik_2 \partial X(1)]^{1 + \frac{t}{2}} e^{ikX(1)} e^{ik_4 X(\infty)} \rangle \\ &= \Gamma\left(-1 - \frac{t}{2}\right) (k_1 k_2)^{1 + \frac{t}{2}} \\ &\sim \Gamma\left(-1 - \frac{t}{2}\right) \left(-\frac{s}{2}\right)^{1 + \frac{t}{2}}. \end{aligned} \quad (13.8)$$

B. Higher spin BPST vertex

1. A spin two state

It was shown [63, 64, 70] that for the $26D$ open bosonic string states of leading order in energy in the Regge limit at mass level $M_2^2 = 2(N - 1)$, $N = \sum_{n,m,l>0} np_n + mq_m + lr_l$ are of the form Eq.(11.75). In this section, we first consider a simple case of a spin two state $\alpha_{-1}^P \alpha_{-1}^P |0\rangle$ corresponding to the vertex $(\partial X^P)^2 e^{ik_2 X}(\omega)$. The four-point amplitude of the

spin two state with three tachyons can be calculated by using the conventional method

$$\begin{aligned}
A^{(q_1=2)} &= \int d\omega \left\langle e^{ik_1 X(0)} (\partial X^P)^2 e^{ik_2 X(\omega)} e^{ik_3 X(1)} e^{ik_4 X(\infty)} \right\rangle \\
&= \int d\omega \omega^{k_1 \cdot k_2} (1-\omega)^{k_2 \cdot k_3} \left[\frac{ie^P \cdot k_1}{-\omega} + \frac{ie^P \cdot k_3}{1-\omega} \right]^2 \\
&= -(e^P \cdot k_1)^2 \Gamma\left(-1 - \frac{t}{2}\right) \left(-\frac{s}{2}\right)^{\frac{t}{2}-1} + 2(e^P \cdot k_1)(e^P \cdot k_3) \Gamma\left(-2 - \frac{t}{2}\right) \left(-\frac{s}{2}\right)^{\frac{t}{2}} \\
&\quad - (e^P \cdot k_3)^2 \Gamma\left(-3 - \frac{t}{2}\right) \left(-\frac{s}{2}\right)^{\frac{t}{2}+1}.
\end{aligned} \tag{13.9}$$

The momenta of the four particles on the scattering plane are

$$k_1 = \left(+\sqrt{p^2 + M_1^2}, -p, 0 \right), \tag{13.10}$$

$$k_2 = \left(+\sqrt{p^2 + M_2^2}, +p, 0 \right), \tag{13.11}$$

$$k_3 = \left(-\sqrt{q^2 + M_3^2}, -q \cos \phi, -q \sin \phi \right), \tag{13.12}$$

$$k_4 = \left(-\sqrt{q^2 + M_4^2}, +q \cos \phi, +q \sin \phi \right) \tag{13.13}$$

where $p \equiv |\tilde{p}|$, $q \equiv |\tilde{q}|$ and $k_i^2 = -M_i^2$. The relevant kinematics in the Regge limit are [63, 64, 70]

$$e^P \cdot k_1 \simeq -\frac{s}{2M_2}, \quad e^P \cdot k_3 \simeq -\frac{\tilde{t}}{2M_2} = -\frac{t - M_2^2 - M_3^2}{2M_2}; \tag{13.14}$$

$$e^L \cdot k_1 \simeq -\frac{s}{2M_2}, \quad e^L \cdot k_3 \simeq -\frac{\tilde{t}'}{2M_2} = -\frac{t + M_2^2 - M_3^2}{2M_2}; \tag{13.15}$$

and

$$e^T \cdot k_1 = 0, \quad e^T \cdot k_3 \simeq -\sqrt{-t} \tag{13.16}$$

where \tilde{t} and \tilde{t}' are related to t by finite mass square terms

$$\tilde{t} = t - M_2^2 - M_3^2, \quad \tilde{t}' = t + M_2^2 - M_3^2. \tag{13.17}$$

By using Eq.(13.14), one easily see that the three terms in Eq.(13.9) share the same order of energy in the Regge limit. We stress that this key observation on the polarizations for higher spin states was not discussed in [61, 146].

One can calculate the OPE in the Regge limit

$$\partial X^P \partial X^P e^{ik_2 X}(w) e^{ik_3 X}(z) \sim |w-z|^{k_2 \cdot k_3} \left[\partial X(z)^P + \frac{ie^P \cdot k_3}{w-z} \right]^2 e^{ikX(z)+ik_2(w-z)\partial X(z)}.$$

This means

$$\partial X^P \partial X^P e^{ik_2 X}(\omega) e^{ik_3 X}(1) \sim (1-\omega)^{k_2 \cdot k_3} \left[\partial X(1)^P - \frac{ie^P \cdot k_3}{1-\omega} \right]^2 e^{ikX(1)-ik_2(1-\omega)\partial X(1)}, k = k_2 + k_3. \quad (13.18)$$

One can carry out the ω integration in Eq.(13.18) at the operator level to obtain the BPST vertex operator

$$\begin{aligned} V_{BPST}^{(q_1=2)} &= \int d\omega (\partial X^P)^2 e^{ik_2 X}(\omega) e^{ik_3 X}(1) \\ &\sim \int d\omega (1-\omega)^{k_2 \cdot k_3} \left[\partial X(1)^P - \frac{ie^P \cdot k_3}{1-\omega} \right]^2 e^{ikX(1)-ik_2(1-\omega)\partial X(1)} \\ &= \partial X(1)^P \partial X(1)^P \int dx x^{k_2 \cdot k_3} e^{ikX(1)-ik_2 x \partial X(1)} \\ &\quad - 2ie^P \cdot k_3 \partial X(1)^P \int dx x^{k_2 \cdot k_3 - 1} e^{ikX(1)-ik_2 x \partial X(1)} \\ &\quad - (e^P \cdot k_3)^2 \int dx x^{k_2 \cdot k_3 - 2} e^{ikX(1)-ik_2 x \partial X(1)} \\ &= \Gamma\left(-1 - \frac{t}{2}\right) [ik_2 \partial X(1)]^{\frac{t}{2}-1} \partial X(1)^P \partial X(1)^P e^{ikX(1)} \\ &\quad - 2ie^P \cdot k_3 \Gamma\left(-2 - \frac{t}{2}\right) [ik_2 \partial X(1)]^{\frac{t}{2}} \partial X(1)^P e^{ikX(1)} \\ &\quad - (e^P \cdot k_3)^2 \Gamma\left(-3 - \frac{t}{2}\right) [ik_2 \partial X(1)]^{\frac{t}{2}+1} e^{ikX(1)} \end{aligned} \quad (13.19)$$

which leads to the same amplitude

$$\begin{aligned} A^{(q_1=2)} &= \left\langle e^{ik_1 X(0)} V_{BPST}^{(q_1=2)} e^{ik_4 X(\infty)} \right\rangle \\ &= \Gamma\left(-1 - \frac{t}{2}\right) \left\langle e^{ik_1 X(0)} [ik_2 \partial X(1)]^{\frac{t}{2}-1} \partial X(1)^P \partial X(1)^P e^{ikX(1)} e^{ik_4 X(\infty)} \right\rangle \\ &\quad - 2ie^P \cdot k_3 \Gamma\left(-2 - \frac{t}{2}\right) \left\langle e^{ik_1 X(0)} [ik_2 \partial X(1)]^{\frac{t}{2}} \partial X(1)^P e^{ikX(1)} e^{ik_4 X(\infty)} \right\rangle \\ &\quad - (e^P \cdot k_3)^2 \Gamma\left(-3 - \frac{t}{2}\right) \left\langle e^{ik_1 X(0)} [ik_2 \partial X(1)]^{\frac{t}{2}+1} e^{ikX(1)} e^{ik_4 X(\infty)} \right\rangle \\ &\sim -(e^P \cdot k_1)^2 \Gamma\left(-1 - \frac{t}{2}\right) \left(-\frac{s}{2}\right)^{\frac{t}{2}-1} + 2(e^P \cdot k_1)(e^P \cdot k_3) \Gamma\left(-2 - \frac{t}{2}\right) \left(-\frac{s}{2}\right)^{\frac{t}{2}} \\ &\quad - (e^P \cdot k_3)^2 \Gamma\left(-3 - \frac{t}{2}\right) \left(-\frac{s}{2}\right)^{\frac{t}{2}+1}. \end{aligned} \quad (13.20)$$

Note that the three terms in Eq.(13.19) lead to the three terms respectively in Eq.(13.20) with the same order of energy in the Regge limit.

2. Higher spin states

We now consider the higher spin state

$$|p_n, q_m\rangle = \prod_{n=1} (\alpha_{-n}^T)^{p_n} \prod_{m=1} (\alpha_{-m}^P)^{q_m} |0\rangle, \quad (13.21)$$

which corresponds to the vertex

$$V_2(\omega) = \left[\prod_{n=1} (\partial^n X^T)^{p_n} \prod_{m=1} (\partial^m X^P)^{q_m} \right] e^{ik_2 X}(\omega). \quad (13.22)$$

The four-point amplitude of the above state with three tachyons was calculated to be (from now on we set $M_2 = M$) [63, 64, 70]

$$\begin{aligned} A^{(p_n, q_m)} &= \int d\omega \langle e^{ik_1 X(0)} V_2(\omega) e^{ik_3 X(1)} e^{ik_4 X(\infty)} \rangle \\ &= \left(-\frac{1}{M}\right)^{q_1} U\left(-q_1, \frac{t}{2} + 2 - q_1, \frac{\tilde{t}}{2}\right) B\left(-1 - \frac{s}{2}, -1 - \frac{t}{2}\right) \\ &\quad \cdot \prod_{n=1} [\sqrt{-t}(n-1)!]^{p_n} \prod_{m=2} \left[\tilde{t}(m-1)! \left(-\frac{1}{2M}\right)\right]^{q_m} \end{aligned} \quad (13.23)$$

$$\sim \left(-\frac{1}{M}\right)^{q_1} U\left(-q_1, \frac{t}{2} + 2 - q_1, \frac{\tilde{t}}{2}\right) \Gamma\left(-1 - \frac{t}{2}\right) \left(-\frac{s}{2}\right)^{1+\frac{t}{2}} \quad (13.24)$$

$$\cdot \prod_{n=1} [\sqrt{-t}(n-1)!]^{p_n} \prod_{m=2} \left[\tilde{t}(m-1)! \left(-\frac{1}{2M}\right)\right]^{q_m} \quad (13.25)$$

where U is the Kummer function of the second kind. One can calculate the OPE in the Regge limit

$$\begin{aligned} &V_2(\omega) e^{ik_3 X(1)} \\ &= \left[\prod_{n=1} (\partial^n X^T)^{p_n} \prod_{m=1} (\partial^m X^P)^{q_m} \right] e^{ik_2 X}(\omega) e^{ik_3 X(1)} \\ &\sim \prod_{n=1} \left[\frac{(n-1)! k_3 \cdot e^T}{(1-\omega)^n} \right]^{p_n} \prod_{m=2} \left[\frac{(m-1)! k_3 \cdot e^P}{(1-\omega)^m} \right]^{q_m} \\ &\quad \cdot \left[\partial X(1) \cdot e^P - \frac{ik_3 \cdot e^P}{1-\omega} \right]^{q_1} (1-\omega)^{k_2 \cdot k_3} e^{ik_X(1) - ik_2(1-\omega)\partial X(1)} \end{aligned} \quad (13.26)$$

$$\begin{aligned} &= \left(\frac{-\tilde{t}}{2M}\right)^{q_1} \prod_{n=1} [\sqrt{-t}(n-1)!]^{p_n} \prod_{m=2} \left[\tilde{t}(m-1)! \left(-\frac{1}{2M}\right)\right]^{q_m} \\ &\quad \cdot \sum_{j=0}^{q_1} \binom{q_1}{j} \left(\frac{2iM_2 \partial X(1) \cdot e^P}{\tilde{t}}\right)^j (1-\omega)^{k_2 \cdot k_3 - N + j} e^{ik_X(1) - ik_2(1-\omega)\partial X(1)} \end{aligned} \quad (13.27)$$

where $N = \sum_{n,m} (np_n + mq_m)$. We can carry out the ω integration in Eq.(13.27) to obtain the BPST vertex operator

$$\begin{aligned}
V_{BPST}^{(p_n; q_m)} &= \int d\omega V_2(\omega) e^{ik_3 X}(1) \\
&\sim \left(\frac{-\tilde{t}}{2M}\right)^{q_1} \prod_{n=1}^{q_1} [\sqrt{-t}(n-1)!]^{p_n} \prod_{m=2} \left[\tilde{t}(m-1)! \left(-\frac{1}{2M}\right) \right]^{q_m} \\
&\cdot \sum_{j=0}^{q_1} \binom{q_1}{j} \left(\frac{2iM_2 \partial X(1) \cdot e^P}{\tilde{t}} \right)^j \int d\omega (1-\omega)^{k_2 \cdot k_3 - N + j} e^{ikX(1) - ik_2(1-\omega)\partial X(1)} \\
&= \left(\frac{-\tilde{t}}{2M}\right)^{q_1} \prod_{n=1}^{q_1} [\sqrt{-t}(n-1)!]^{p_n} \prod_{m=2} \left[\tilde{t}(m-1)! \left(-\frac{1}{2M}\right) \right]^{q_m} \\
&\cdot \sum_{j=0}^{q_1} \binom{q_1}{j} \left(\frac{2iM \partial X(1) \cdot e^P}{\tilde{t}} \right)^j \int dx x^{k_2 \cdot k_3 - N + j} e^{ikX(1) - ik_2 x \partial X(1)} \\
&= \left(\frac{-\tilde{t}}{2M}\right)^{q_1} \prod_{n=1}^{q_1} [\sqrt{-t}(n-1)!]^{p_n} \prod_{m=2} \left[\tilde{t}(m-1)! \left(-\frac{1}{2M}\right) \right]^{q_m} \\
&\cdot \sum_{j=0}^{q_1} \binom{q_1}{j} \left(\frac{2iM \partial X(1) \cdot e^P}{\tilde{t}} \right)^j \Gamma\left(-1 - \frac{t}{2} + j\right) [ik_2 \cdot \partial X(1)]^{1 + \frac{t}{2} - j} e^{ikX(1)}. \quad (13.28)
\end{aligned}$$

One notes that, in Eq.(13.28), $M \partial X(1) \cdot e^P = k_2 \cdot \partial X(1)$ and the summation over j can be simplified. The BPST vertex operator can be further reduced to

$$\begin{aligned}
V_{BPST}^{(p_n; q_m)} &= \left(\frac{-\tilde{t}}{2M_2}\right)^{q_1} \prod_{n=1}^{q_1} [\sqrt{-t}(n-1)!]^{p_n} \prod_{m=2} \left[\tilde{t}(m-1)! \left(-\frac{1}{2M}\right) \right]^{q_m} \\
&\cdot \sum_{j=0}^{q_1} \binom{q_1}{j} \left(\frac{2}{\tilde{t}}\right)^j \left(-1 - \frac{t}{2}\right)_j \Gamma\left(-1 - \frac{t}{2}\right) [ik_2 \cdot \partial X(1)]^{1 + \frac{t}{2}} e^{ikX(1)} \\
&= \left(\frac{-1}{M}\right)^{q_1} \prod_{n=1}^{q_1} [\sqrt{-t}(n-1)!]^{p_n} \prod_{m=2} \left[\tilde{t}(m-1)! \left(-\frac{1}{2M}\right) \right]^{q_m} \\
&\cdot U\left(-q_1, \frac{t}{2} + 2 - q_1, \frac{\tilde{t}}{2}\right) \Gamma\left(-1 - \frac{t}{2}\right) [ik_2 \cdot \partial X(1)]^{1 + \frac{t}{2}} e^{ikX(1)} \quad (13.29)
\end{aligned}$$

where we have used

$$\sum_{j=0}^l \binom{l}{j} \left(\frac{2}{\tilde{t}}\right)^j \left(-1 - \frac{t}{2}\right)_j = 2^l (\tilde{t})^{-l} U\left(-l, \frac{t}{2} + 2 - l, \frac{\tilde{t}}{2}\right). \quad (13.30)$$

One notes that the exponent of $[ik_2 \cdot \partial X(1)]^{1 + \frac{t}{2}}$ in Eq.(13.29) is mass level N independent. This is related to the fact that the well known $\sim s^{\alpha(t)}$ power-law behavior of the four tachyon string scattering amplitude in the RR can be extended to arbitrary higher string states and is mass level independent as can be seen from Eq.(13.24). This interesting result was first

pointed out in section XI.C [63] and will be crucial to derive inter-mass level recurrence relations among BPST vertex operators to be discussed later.

The BPST vertex operator in Eq.(13.29) leads to exactly the same amplitude as in Eq.(13.25).

C. Recurrence relations

For any confluent hypergeometric function $U(a, c, x)$ with parameters (a, c) the four functions with parameters $(a - 1, c)$, $(a + 1, c)$, $(a, c - 1)$ and $(a, c + 1)$ are called the contiguous functions. Recurrence relation exists between any such function and any two of its contiguous functions. There are six recurrence relations [71]

$$U(a - 1, c, x) - (2a - c + x)U(a, c, x) + a(1 + a - c)U(a + 1, c, x) = 0, \quad (13.31)$$

$$(c - a - 1)U(a, c - 1, x) - (x + c - 1)U(a, c, x) + xU(a, c + 1, x) = 0, \quad (13.32)$$

$$U(a, c, x) - aU(a + 1, c, x) - U(a, c - 1, x) = 0, \quad (13.33)$$

$$(c - a)U(a, c, x) + U(a - 1, c, x) - xU(a, c + 1, x) = 0, \quad (13.34)$$

$$(a + x)U(a, c, x) - xU(a, c + 1, x) + a(c - a - 1)U(a + 1, c, x) = 0, \quad (13.35)$$

$$(a + x - 1)U(a, c, x) - U(a - 1, c, x) + (1 + a - c)U(a, c - 1, x) = 0. \quad (13.36)$$

From any two of these six relations the remaining four recurrence relations can be deduced.

The confluent hypergeometric function $U(a, c, x)$ with parameters $(a \pm m, c \pm n)$ for $m, n = 0, 1, 2, \dots$ are called associated functions. Again it can be shown that there exist relations between any three associated functions, so that any confluent hypergeometric function can be expressed in terms of any two of its associated functions.

Recently it was shown [70] that Recurrence relations exist among higher spin Regge string scattering amplitudes of different string states. This was discussed in section XI.D. The key to derive these relations was to use recurrence relations and addition theorem of Kummer functions. In view of the form of higher spin BPST vertex operators in Eq.(13.29), one can easily calculate recurrence relations among higher spin BPST vertex operators. By using the recurrence relation of Kummer functions [70], for example,

$$U\left(-2, \frac{t}{2}, \frac{t}{2}\right) + \left(\frac{t}{2} + 1\right)U\left(-1, \frac{t}{2}, \frac{t}{2}\right) - \frac{t}{2}U\left(-1, \frac{t}{2} + 1, \frac{t}{2}\right) = 0, \quad (13.37)$$

one can obtain the following recurrence relation among BPST vertex operators at mass level $M^2 = 2$ [72]

$$M\sqrt{-t}V_{BPST}^{(q_1=2)} - \frac{t}{2}V_{BPST}^{(p_1=1, q_1=1)} = 0. \quad (13.38)$$

Rather than constant coefficients in the RR Regge stringy Ward identities derived in [70], the coefficients of this recurrence relation Eq.(13.38) among BPST vertex operators are kinematic variable dependent, similar to BCJ relations among field theory amplitudes [37, 40, 123–125]. The recurrence relation among BPST vertex operators in Eq.(13.38) leads to the recurrence relation among Regge string scattering amplitudes [70]

$$M\sqrt{-t}A^{(q_1=2)} - \frac{t}{2}A^{(p_1=1, q_1=1)} = 0, \quad (13.39)$$

which is the same with Eq.(11.147).

D. More general recurrence relations

To derive more general recurrence relations, we need to calculate BPST vertex operators corresponding to the general higher spin states in Eq.(11.75). We first calculate the BPST vertex operator correspond to the state

$$|p_n, r_l\rangle = \prod_{n=1} (\alpha_{-n}^T)^{p_n} \prod_{m=1} (\alpha_{-l}^L)^{r_l} |0\rangle. \quad (13.40)$$

The calculation is very similar to that of Eq.(13.21) up to some modification. One can easily get that Eq.(13.28) is now replaced by

$$\begin{aligned} V_{BPST}^{(p_n; r_l)} &= \left(\frac{-\tilde{t}'}{2M}\right)^{r_1} \prod_{n=1} [\sqrt{-t}(n-1)!]^{p_n} \prod_{l=2} \left[\tilde{t}'(l-1)! \left(-\frac{1}{2M}\right)\right]^{r_l} \\ &\cdot \sum_{j=0}^{r_1} \binom{r_1}{j} \left(\frac{2iM\partial X(1) \cdot e^L}{\tilde{t}'}\right)^j \Gamma\left(-1 - \frac{t}{2} + j\right) [ik_2 \cdot \partial X(1)]^{1+\frac{t}{2}-j} e^{ikX(1)}. \end{aligned} \quad (13.41)$$

One notes that, in Eq.(13.41), $M\partial X(1) \cdot e^L \neq k_2 \cdot \partial X(1)$ and, in contrast to Eq.(13.28), the two factors with exponents j and $-j$ do not cancel out. The BPST vertex operator for this case thus reduces to

$$\begin{aligned} V_{BPST}^{(p_n; r_l)} &= \left(\frac{-1}{M}\right)^{r_1} \prod_{n=1} [\sqrt{-t}(n-1)!]^{p_n} \prod_{l=2} \left[\tilde{t}'(l-1)! \left(-\frac{1}{2M}\right)\right]^{r_l} \\ &\cdot U\left(-r_1, \frac{t}{2} + 2 - r_1, \frac{\tilde{t}' e^P \cdot \partial X(1)}{2 e^L \cdot \partial X(1)}\right) \Gamma\left(-1 - \frac{t}{2}\right) [ik_2 \cdot \partial X(1)]^{1+\frac{t}{2}} e^{ikX(1)}. \end{aligned} \quad (13.42)$$

The BPST vertex operator in Eq.(13.42) leads to the amplitude

$$A^{(p_n, r_l)} = \left(-\frac{1}{M}\right)^{r_1} U\left(-r_1, \frac{t}{2} + 2 - r_1, \frac{\tilde{t}'}{2}\right) \Gamma\left(-1 - \frac{t}{2}\right) \left(-\frac{s}{2}\right)^{1+\frac{t}{2}} \cdot \prod_{n=1} [\sqrt{-t}(n-1)!]^{p_n} \prod_{l=2} \left[\tilde{t}'(l-1)! \left(-\frac{1}{2M}\right)\right]^{r_l}, \quad (13.43)$$

which is consistent with the one calculated in [63, 64, 70]. Note that the contribution of $\frac{e^P \cdot \partial X(1)}{e^L \cdot \partial X(1)}$ in the correlation function reduces to 1 in the Regge limit by using first equations of Eq.(13.14) and Eq.(13.15). One sees that Eq.(13.43) can be obtained from Eq.(13.25) by doing the replacement $\tilde{t} \rightarrow \tilde{t}'$.

We are now ready to calculate the BPST vertex operator corresponding to the most general Regge state in Eq.(11.75). Similar to the RR amplitude calculated in Eq.(11.86) and Eq.(11.87) [70], the BPST vertex operator can be expressed in two equivalent forms

$$V_{BPST}^{(p_n; q_m; r_l)} = \prod_{n=1} [(n-1)! \sqrt{-t}]^{p_n} \cdot \prod_{m=1} \left[-(m-1)! \frac{\tilde{t}}{2M}\right]^{q_m} \cdot \prod_{l=2} \left[(l-1)! \frac{\tilde{t}'}{2M}\right]^{r_l} \cdot \left(\frac{1}{M}\right)^{r_1} \Gamma\left(-1 - \frac{t}{2}\right) [ik_2 \cdot \partial X(1)]^{1+\frac{t}{2}} e^{ikX(1)} \cdot \sum_{i=0}^{q_1} \binom{q_1}{i} \left(\frac{2}{\tilde{t}}\right)^i \left(-\frac{t}{2} - 1\right)_i U\left(-r_1, \frac{t}{2} + 2 - i - r_1, \frac{\tilde{t}'}{2} \frac{e^P \cdot \partial X(1)}{e^L \cdot \partial X(1)}\right) \quad (13.44)$$

$$= \prod_{n=1} [(n-1)! \sqrt{-t}]^{p_n} \cdot \prod_{m=2} \left[-(m-1)! \frac{\tilde{t}}{2M}\right]^{q_m} \cdot \prod_{l=1} \left[(l-1)! \frac{\tilde{t}'}{2M}\right]^{r_l} \cdot \left(-\frac{1}{M}\right)^{q_1} \Gamma\left(-1 - \frac{t}{2}\right) [ik_2 \cdot \partial X(1)]^{1+\frac{t}{2}} e^{ikX(1)} \cdot \sum_{j=0}^{r_1} \binom{r_1}{j} \left(\frac{2}{\tilde{t}'} \frac{e^L \cdot \partial X(1)}{e^P \cdot \partial X(1)}\right)^j \left(-\frac{t}{2} - 1\right)_j U\left(-q_1, \frac{t}{2} + 2 - j - q_1, \frac{\tilde{t}}{2}\right). \quad (13.45)$$

Either form Eq.(13.44) or Eq.(13.45) of the above BPST vertex operator leads consistently

to the amplitude calculated previously in Eq.(11.86) and Eq.(11.87) and is re-listed here [70]

$$\begin{aligned}
A^{(p_n, q_m; r_l)} &= \prod_{n=1} [(n-1)! \sqrt{-t}]^{p_n} \cdot \prod_{m=1} \left[-(m-1)! \frac{\tilde{t}}{2M} \right]^{q_m} \cdot \prod_{l=2} \left[(l-1)! \frac{\tilde{t}'}{2M} \right]^{r_l} \\
&\cdot \left(\frac{1}{M} \right)^{r_1} \Gamma \left(-1 - \frac{t}{2} \right) \left(-\frac{s}{2} \right)^{1+\frac{t}{2}} \\
&\cdot \sum_{i=0}^{q_1} \binom{q_1}{i} \left(\frac{2}{\tilde{t}} \right)^i \left(-\frac{t}{2} - 1 \right)_i U \left(-r_1, \frac{t}{2} + 2 - i - r_1, \frac{\tilde{t}'}{2} \right)
\end{aligned} \tag{13.46}$$

$$\begin{aligned}
&= \prod_{n=1} [(n-1)! \sqrt{-t}]^{p_n} \cdot \prod_{m=2} \left[-(m-1)! \frac{\tilde{t}}{2M} \right]^{q_m} \cdot \prod_{l=1} \left[(l-1)! \frac{\tilde{t}'}{2M} \right]^{r_l} \\
&\cdot \left(-\frac{1}{M} \right)^{q_1} \Gamma \left(-1 - \frac{t}{2} \right) \left(-\frac{s}{2} \right)^{1+\frac{t}{2}} \\
&\cdot \sum_{j=0}^{r_1} \binom{r_1}{j} \left(\frac{2}{\tilde{t}'} \right)^j \left(-\frac{t}{2} - 1 \right)_j U \left(-q_1, \frac{t}{2} + 2 - j - q_1, \frac{\tilde{t}}{2} \right).
\end{aligned} \tag{13.47}$$

One can now derive more general recurrence relations among BPST vertex operators. As an example, the three BPST vertex operators $V_{BPST}^{q_1=3}$, $V_{BPST}^{p_1=1, q_1=2}$ and $V_{BPST}^{q_1=2, r_1=1}$ can be calculated by using Eq.(13.45) to be

$$V_{BPST}^{(q_1=3)} = \left(-\frac{1}{M} \right)^3 \Gamma \left(-1 - \frac{t}{2} \right) [ik_2 \cdot \partial X(1)]^{1+\frac{t}{2}} e^{ikX(1)} U \left(-3, \frac{t}{2} - 1, \frac{t}{2} - 1 \right), \tag{13.48}$$

$$V_{BPST}^{(p_1=1, q_1=2)} = \left(-\frac{1}{M} \right)^2 \sqrt{-t} \Gamma \left(-1 - \frac{t}{2} \right) [ik_2 \cdot \partial X(1)]^{1+\frac{t}{2}} e^{ikX(1)} U \left(-2, \frac{t}{2}, \frac{t}{2} - 1 \right), \tag{13.49}$$

$$\begin{aligned}
V_{BPST}^{(q_1=2, r_1=1)} &= \frac{t+6}{2M} \left(-\frac{1}{M} \right)^2 \Gamma \left(-1 - \frac{t}{2} \right) [ik_2 \cdot \partial X(1)]^{1+\frac{t}{2}} e^{ikX(1)} \\
&\left[U \left(-2, \frac{t}{2}, \frac{t}{2} - 1 \right) + \frac{2}{t+6} \left(-\frac{t}{2} - 1 \right) U \left(-2, \frac{t}{2} - 1, \frac{t}{2} - 1 \right) \frac{e^L \cdot \partial X(1)}{e^P \cdot \partial X(1)} \right].
\end{aligned} \tag{13.50}$$

The recurrence relation among Kummer functions derived from Eq.(13.34) [70]

$$U \left(-3, \frac{t}{2} - 1, \frac{t}{2} - 1 \right) + \left(\frac{t}{2} + 1 \right) U \left(-2, \frac{t}{2} - 1, \frac{t}{2} - 1 \right) - \left(\frac{t}{2} - 1 \right) U \left(-2, \frac{t}{2}, \frac{t}{2} - 1 \right) = 0 \tag{13.51}$$

leads to the following recurrence relation among BPST vertex operators at mass level $M^2 = 4$

[72]

$$M\sqrt{-t}e^L \cdot \partial X(1) V_{BPST}^{q_1=3} + M\sqrt{-t}e^P \cdot \partial X(1) V_{BPST}^{q_1=2, r_1=1} - \left[\left(\frac{t}{2} + 3 \right) e^P \cdot \partial X(1) - \left(\frac{t}{2} - 1 \right) e^L \cdot \partial X(1) \right] V_{BPST}^{p_1=1, q_1=2} = 0. \quad (13.52)$$

In addition to the t dependence, the coefficients of the recurrence relation in Eq.(13.52) are operator dependent. The recurrence relation among BPST vertex operators in Eq.(13.52) leads to the recurrence relation among Regge string scattering amplitudes [70]

$$M\sqrt{-t}A^{(q_1=3)} - 4A^{(p_1=1, q_1=2)} + M\sqrt{-t}A^{(q_1=2, r_1=1)} = 0, \quad (13.53)$$

which is the same with Eq.(11.152).

For the next example, we construct an inter-mass level recurrence relation for BPST vertex operators at mass level $M^2 = 2, 4$. We begin with the addition theorem of Kummer function [71]

$$U(a, c, x + y) = \sum_{k=0}^{\infty} \frac{1}{k!} (a)_k (-1)^k y^k U(a + k, c + k, x) \quad (13.54)$$

which terminates to a finite sum for a non-positive integer a . By taking, for example, $a = -1, c = \frac{t}{2} + 1, x = \frac{t}{2} - 1$ and $y = 1$, the theorem gives [70]

$$U\left(-1, \frac{t}{2} + 1, \frac{t}{2}\right) - U\left(-1, \frac{t}{2} + 1, \frac{t}{2} - 1\right) - U\left(0, \frac{t}{2} + 2, \frac{t}{2} - 1\right) = 0. \quad (13.55)$$

Eq.(13.55) leads to an inter-mass level recurrence relation among BPST vertex operators [72]

$$M(2)(t + 6)V_{BPST}^{(p_1=1, q_1=1)} - 2M(4)^2\sqrt{-t}V_{BPST}^{(q_1=1, r_2=1)} + 2M(4)V_{BPST}^{(p_1=1, r_2=1)} = 0 \quad (13.56)$$

where masses $M(2) = \sqrt{2}, M(4) = \sqrt{4} = 2$, and $V_{BPST}^{p_1=1, q_1=1}$ is a BPST vertex operator at mass level $M^2 = 2$, and $V_{BPST}^{q_1=1, r_2=1}, V_{BPST}^{p_1=1, r_2=1}$ are BPST vertex operators at mass levels $M^2 = 4$ respectively. In deriving Eq.(13.56), it is important to use the fact that the exponent of $[ik_2 \cdot \partial X(1)]^{1+\frac{t}{2}}$ in the BPST vertex operator in Eq.(13.45) is mass level N independent as mentioned in the paragraph after Eq.(13.30). The recurrence relation among BPST vertex operators in Eq.(13.56) leads to the recurrence relation among Regge string scattering amplitudes [70]

$$M(2)(t + 6)A^{(p_1=1, q_1=1)} - 2M(4)^2\sqrt{-t}A^{(q_1=1, r_2=1)} + 2M(4)A^{(p_1=1, r_2=1)} = 0, \quad (13.57)$$

which is the same with Eq.(11.155).

In section XI.D.2 [70], it was shown that, at each fixed mass level, each Kummer function in the summation of Eq.(13.47) can be expressed in terms of Regge string scattering amplitudes $A^{(p_n, q_m; r_l)}$ at the same mass level. Moreover, although for general values of a , the best one can obtain from recurrence relations of Kummer function $U(a, c, x)$ is to express any Kummer function in terms of any two of its associated function, for non-positive integer values of a in the RR string amplitude case however, $U(a, c, x)$ can be fixed up to an overall factor by using Kummer function recurrence relations [70]. As a result, all Regge string scattering amplitudes can be algebraically solved by Kummer function recurrence relations up to multiplicative factors. An important application of the above properties is the construction of an infinite number of recurrence relations among Regge string scattering amplitudes. One can use the recurrence relations of Kummer functions Eq.(13.31) to Eq.(13.36) to systematically construct recurrence relations among Regge string scattering amplitudes.

In view of the form of BPST vertex operators calculated in Eq.(13.45), one can similarly solve [70] all Kummer functions $U(a, c, x)$ in Eq.(13.45) in terms of BPST vertex operators and use the recurrence relations of Kummer functions Eq.(13.31) to Eq.(13.36) to systematically construct an infinite number of recurrence relations among BPST vertex operators. Moreover, the forms of all BPST vertex operators can be fixed by these recurrence relations up to multiplicative factors. These recurrence relations among BPST vertex operators are dual to linear relations or symmetries among high energy fixed angle string scattering amplitudes discovered previously [26–28, 30, 31, 44].

We illustrate the prescription here to construct other examples of recurrence relations among BPST vertex operators at mass level $M^2 = 4$. Generalization to arbitrary mass levels will be given in the next section. There are 22 BPST vertex operators for the mass level $M^2 = 4$. We first consider the group of BPST vertex operators with $q_1 = 0$, $(V_{BPST}^{TTT}, V_{BPST}^{LTT}, V_{BPST}^{LLT}, V_{BPST}^{LLL})$ [70]. The corresponding r_1 for each BPST vertex operator are $(0, 1, 2, 3)$. Here we use a new notation for BPST vertex operator, for example, $V_{BPST}^{LLT} \equiv V_{BPST}^{(p_1=1, r_1=2)}$, $V_{BPST}^{LTT} = V_{BPST}^{(p_1=1, r_2=1)}$ and $V_{BPST}^{TL} = V_{BPST}^{(p_2=1, r_1=1)}$ etc. By using

Eq.(13.45), one can easily calculate that

$$V_{BPST}^{TTT} = (\sqrt{-t})^3 \Gamma\left(-1 - \frac{t}{2}\right) [ik_2 \cdot \partial X(1)]^{1+\frac{t}{2}} e^{ikX(1)} U\left(0, \frac{t}{2} + 2, \frac{t}{2} - 1\right), \quad (13.58)$$

$$V_{BPST}^{LTT} = \frac{t+6}{2M} (\sqrt{-t})^2 \Gamma\left(-1 - \frac{t}{2}\right) [ik_2 \cdot \partial X(1)]^{1+\frac{t}{2}} e^{ikX(1)} \cdot \left[U\left(0, \frac{t}{2} + 2, \frac{t}{2} - 1\right) + \frac{2}{t+6} \left(-\frac{t}{2} - 1\right) U\left(0, \frac{t}{2} + 1, \frac{t}{2} - 1\right) \frac{e^L \cdot \partial X(1)}{e^P \cdot \partial X(1)} \right], \quad (13.59)$$

$$V_{BPST}^{LLT} = \left(\frac{t+6}{2M}\right)^2 (\sqrt{-t}) \Gamma\left(-1 - \frac{t}{2}\right) [ik_2 \cdot \partial X(1)]^{1+\frac{t}{2}} e^{ikX(1)} \cdot \left[U\left(0, \frac{t}{2} + 2, \frac{t}{2} - 1\right) + \frac{4}{t+6} \left(-\frac{t}{2} - 1\right) U\left(0, \frac{t}{2} + 1, \frac{t}{2} - 1\right) \frac{e^L \cdot \partial X(1)}{e^P \cdot \partial X(1)} + \left(\frac{2}{t+6}\right)^2 \left(-\frac{t}{2} - 1\right) \left(-\frac{t}{2}\right) U\left(0, \frac{t}{2}, \frac{t}{2} - 1\right) \left[\frac{e^L \cdot \partial X(1)}{e^P \cdot \partial X(1)}\right]^2 \right], \quad (13.60)$$

$$V_{BPST}^{LLL} = \left(\frac{t+6}{2M}\right)^3 \Gamma\left(-1 - \frac{t}{2}\right) [ik_2 \cdot \partial X(1)]^{1+\frac{t}{2}} e^{ikX(1)} \cdot \left[U\left(0, \frac{t}{2} + 2, \frac{t}{2} - 1\right) + \frac{6}{t+6} \left(-\frac{t}{2} - 1\right) U\left(0, \frac{t}{2} + 1, \frac{t}{2} - 1\right) \frac{e^L \cdot \partial X(1)}{e^P \cdot \partial X(1)} + 3\left(\frac{2}{t+6}\right)^2 \left(-\frac{t}{2} - 1\right) \left(-\frac{t}{2}\right) U\left(0, \frac{t}{2}, \frac{t}{2} - 1\right) \left[\frac{e^L \cdot \partial X(1)}{e^P \cdot \partial X(1)}\right]^2 + \left(\frac{2}{t+6}\right)^3 \left(-\frac{t}{2} - 1\right) \left(-\frac{t}{2}\right) \left(-\frac{t}{2} + 1\right) U\left(0, \frac{t}{2} - 1, \frac{t}{2} - 1\right) \left[\frac{e^L \cdot \partial X(1)}{e^P \cdot \partial X(1)}\right]^3 \right]. \quad (13.61)$$

From the above equations, one can easily see that $U\left(0, \frac{t}{2} + 2, \frac{t}{2} - 1\right)$ can be expressed in terms of V_{BPST}^{TTT} , $U\left(0, \frac{t}{2} + 1, \frac{t}{2} - 1\right)$ can be expressed in terms of $(V_{BPST}^{TTT}, V_{BPST}^{LTT})$, $U\left(0, \frac{t}{2}, \frac{t}{2} - 1\right)$ can be expressed in terms of $(V_{BPST}^{TTT}, V_{BPST}^{LTT}, V_{BPST}^{LLT})$, and finally $U\left(0, \frac{t}{2} - 1, \frac{t}{2} - 1\right)$ can be expressed in terms of $(V_{BPST}^{TTT}, V_{BPST}^{LTT}, V_{BPST}^{LLT}, V_{BPST}^{LLL})$. We have

$$U\left(0, \frac{t}{2} + 2, \frac{t}{2} - 1\right) = \Omega^{-1} (\sqrt{-t})^{-3} V_{BPST}^{TTT}, \quad (13.62)$$

$$U\left(0, \frac{t}{2} + 1, \frac{t}{2} - 1\right) = \Omega^{-1} (\sqrt{-t})^{-3} \frac{t+6}{t+2} \left[\frac{e^P \cdot \partial X(1)}{e^L \cdot \partial X(1)} \right] \cdot \left[V_{BPST}^{TTT} - \frac{2M}{t+6} \sqrt{-t} V_{BPST}^{LTT} \right], \quad (13.63)$$

$$U\left(0, \frac{t}{2}, \frac{t}{2} - 1\right) = \Omega^{-1} (\sqrt{-t})^{-3} \frac{(t+6)^2}{t(t+2)} \left[\frac{e^P \cdot \partial X(1)}{e^L \cdot \partial X(1)} \right]^2 \cdot \left[V_{BPST}^{TTT} - 2 \frac{2M}{t+6} \sqrt{-t} V_{BPST}^{LTT} + \left(\frac{2M}{t+6} \sqrt{-t} \right)^2 V_{BPST}^{LLT} \right], \quad (13.64)$$

$$U\left(0, \frac{t}{2} - 1, \frac{t}{2} - 1\right) = \Omega^{-1} (\sqrt{-t})^{-3} \frac{(t+6)^3}{t(t^2-4)} \left[\frac{e^P \cdot \partial X(1)}{e^L \cdot \partial X(1)} \right]^3 \cdot \left[V_{BPST}^{TTT} - 3 \frac{2M}{t+6} \sqrt{-t} V_{BPST}^{LTT} + 3 \left(\frac{2M}{t+6} \sqrt{-t} \right)^2 V_{BPST}^{LLT} - \left(\frac{2M}{t+6} \sqrt{-t} \right)^3 V_{BPST}^{LLL} \right] \quad (13.65)$$

where $\Omega \equiv \Gamma\left(-1 - \frac{t}{2}\right) [ik_2 \cdot \partial X(1)]^{1+\frac{t}{2}} e^{ikX(1)}$. To derive an example of recurrence relation, one notes that Eq.(13.32) gives

$$\frac{t}{2}U\left(0, \frac{t}{2}, \frac{t}{2} - 1\right) - (t-1)U\left(0, \frac{t}{2} + 1, \frac{t}{2} - 1\right) + \left(\frac{t}{2} - 1\right)U\left(0, \frac{t}{2} + 2, \frac{t}{2} - 1\right) = 0, \quad (13.66)$$

which leads to the recurrence relation among BPST vertex operators

$$\begin{aligned} & \left[\left(\frac{t}{2} - 1\right) - \frac{(t-1)(t+6)}{t+2} \frac{e^P \cdot \partial X(1)}{e^L \cdot \partial X(1)} + \frac{(t+6)^2}{2(t+2)} \left[\frac{e^P \cdot \partial X(1)}{e^L \cdot \partial X(1)} \right]^2 \right] V_{BPST}^{TTT} \\ & + \left[\frac{(t-1)}{t+2} \frac{e^P \cdot \partial X(1)}{e^L \cdot \partial X(1)} - \frac{(t+6)}{(t+2)} \left[\frac{e^P \cdot \partial X(1)}{e^L \cdot \partial X(1)} \right]^2 \right] (2M\sqrt{-t}) V_{BPST}^{LTT} \\ & + \left[\frac{1}{2(t+2)} \left[\frac{e^P \cdot \partial X(1)}{e^L \cdot \partial X(1)} \right]^2 \right] (2M\sqrt{-t})^2 V_{BPST}^{LLT} = 0. \quad (13.67) \end{aligned}$$

Again one can use Eq.(13.67) to deduce recurrence relation among Regge string scattering amplitudes [72]

$$(t+22)A^{(p_1=3)} - 14M\sqrt{-t}A^{(p_1=2, r_1=1)} + 2M^2(\sqrt{-t})^2 A^{(p_1=1, r_1=2)} = 0. \quad (13.68)$$

Other recurrence relations of Kummer functions can be used to derive more recurrence relations among BPST vertex operators. For example, Eq.(13.32) gives a recurrence relation of $U\left(0, \frac{t}{2} + 1, \frac{t}{2} - 1\right)$ and its associated functions $U\left(0, \frac{t}{2} - 1, \frac{t}{2} - 1\right)$ and $U\left(0, \frac{t}{2} + 2, \frac{t}{2} - 1\right)$

$$tU\left(0, \frac{t}{2} - 1, \frac{t}{2} - 1\right) - (3t-4)U\left(0, \frac{t}{2} + 1, \frac{t}{2} - 1\right) + 2(t-2)U\left(0, \frac{t}{2} + 2, \frac{t}{2} - 1\right) = 0, \quad (13.69)$$

which leads to the recurrence relation among BPST vertex operators

$$\begin{aligned} & \left[2(t-2) - \frac{(3t-4)(t+6)}{t+2} \frac{e^P \cdot \partial X(1)}{e^L \cdot \partial X(1)} + \frac{(t+6)^3}{(t^2-4)} \left[\frac{e^P \cdot \partial X(1)}{e^L \cdot \partial X(1)} \right]^3 \right] V_{BPST}^{TTT} \\ & + \left[\frac{(3t-4)}{t+2} \frac{e^P \cdot \partial X(1)}{e^L \cdot \partial X(1)} - 3 \frac{(t+6)^2}{(t^2-4)} \left[\frac{e^P \cdot \partial X(1)}{e^L \cdot \partial X(1)} \right]^3 \right] (2M\sqrt{-t}) V_{BPST}^{LTT} \\ & + \left[\frac{3(t+6)}{(t^2-4)} \left[\frac{e^P \cdot \partial X(1)}{e^L \cdot \partial X(1)} \right]^3 \right] (2M\sqrt{-t})^2 V_{BPST}^{LLT} \\ & - \left[\frac{1}{(t^2-4)} \left[\frac{e^P \cdot \partial X(1)}{e^L \cdot \partial X(1)} \right]^3 \right] (2M\sqrt{-t})^3 V_{BPST}^{LLL} = 0. \quad (13.70) \end{aligned}$$

one can use Eq.(13.70) to deduce recurrence relation among Regge string scattering ampli-

tudes [72]

$$(3t^2 + 76t + 92)A^{(p_1=3)} - 2(23t + 50)M\sqrt{-t}A^{(p_1=2, r_1=1)} + 6M^2(t + 6)(\sqrt{-t})^2A^{(p_1=1, r_1=2)} - 4M^3(\sqrt{-t})^3A^{(r_1=3)} = 0. \quad (13.71)$$

Similarly, we can consider groups of BPST vertex operators $(V_{BPST}^{PT}, V_{BPST}^{PL}), (V_{BPST}^{LT}, V_{BPST}^{LL})$ and $(V_{BPST}^{TT}, V_{BPST}^{TL})$ with $q_1 = 0$; group of BPST vertex operators $(V_{BPST}^{PTT}, V_{BPST}^{PLT}, V_{BPST}^{PLL})$ with $q_1 = 1$ and group of BPST vertex operators $(V_{BPST}^{PPT}, V_{BPST}^{PPL})$ with $q_1 = 2$. All the remaining 7 BPST vertex operators are with $r_1 = 0$, and each BPST vertex operators contains only one Kummer function. Thus all Kummer functions involved at mass level $M^2 = 4$ can be algebraically solved and expressed in terms of BPST vertex operators. One can then use recurrence relations of Kummer functions to derive more recurrence relations among BPST vertex operators.

E. Arbitrary mass levels

In this section, we solve the Kummer functions in terms of the highest spin string states scattering amplitudes for arbitrary mass levels. The highest spin string states at the mass level $M^2 = 2(N - 1)$ are defined as

$$|N - q_1 - r_1, q_1, r_1\rangle = (\alpha_{-1}^T)^{N-q_1-r_1} (\alpha_{-1}^P)^{q_1} (\alpha_{-1}^L)^{r_1} |0, k\rangle \quad (13.72)$$

where only α_{-1} operator appears. The highest spin string states BPST vertex operators can be easily obtained from Eq.(13.45) as

$$\begin{aligned} (V^T)^{N-q_1-r_1} (V^P)^{q_1} (V^L)^{r_1} &\equiv V_{BPST}^{(N-q_1-r_1, q_1, r_1)} \\ &= \Gamma\left(-\frac{t}{2} - 1\right) [ik_2 \cdot \partial X(1)]^{1+\frac{t}{2}} e^{ikX(1)} (\sqrt{-t})^{N-q_1-r_1} \left(-\frac{1}{M}\right)^{q_1} \left(\frac{\tilde{t}'}{2M}\right)^{r_1} \\ &\cdot \sum_{j=0}^{r_1} \binom{r_1}{j} \left(\frac{2}{\tilde{t}'} \frac{e^L \cdot \partial X(1)}{e^P \cdot \partial X(1)}\right)^j \left(-\frac{t}{2} - 1\right)_j U\left(-q_1, \frac{t}{2} + 2 - j - q_1, \frac{\tilde{t}}{2}\right). \end{aligned} \quad (13.73)$$

In view of the form of Eq.(13.65), we can solve the Kummer function from Eq.(13.73) and express it in terms of the highest spin BPST vertex operators as

$$U\left(-q_1, \frac{t}{2} + 2 - q_1 - r_1, \frac{\tilde{t}}{2}\right) = \frac{\Gamma\left(-\frac{t}{2} - 1\right)}{\left(-\frac{t}{2} - 1\right)_{r_1}} [ik_2 \cdot \partial X(1)]^{1+\frac{t}{2}} e^{ikX(1)} \\ \cdot (-MV^P)^{q_1} \left(\frac{V^T}{\sqrt{-t}}\right)^{N-q_1} \left[\frac{e^P \cdot \partial X(1)}{e^L \cdot \partial X(1)} \left(\sqrt{-t}M \frac{V^L}{V^T} - \frac{\tilde{t}'}{2}\right)\right]^{r_1}. \quad (13.74)$$

Putting the Kummer functions Eq.(13.74) into the recurrence relations Eqs.(13.31-13.36), we can then obtain recurrence relations among BPST vertex operators.

Let us consider, for example, the recurrence relation

$$(c - a - 1)U(a, c - 1, x) - (x + c - 1)U(a, c, x) + xU(a, c + 1, x) = 0. \quad (13.75)$$

With

$$a = -q_1, c = \frac{t}{2} + 1 - q_1 - r_1, x = \frac{\tilde{t}}{2} = \frac{t - M^2 + 2}{2}, \quad (13.76)$$

the above recurrence relation becomes

$$\left(\frac{t}{2} - r_1\right)U\left(-q_1, \frac{t}{2} - q_1 - r_1, \frac{\tilde{t}}{2}\right) \\ - \left(\frac{\tilde{t}}{2} + \frac{t}{2} - q_1 - r_1\right)U\left(-q_1, \frac{t}{2} + 1 - q_1 - r_1, \frac{\tilde{t}}{2}\right) \\ + \frac{\tilde{t}}{2}U\left(-q_1, \frac{t}{2} + 2 - q_1 - r_1, \frac{\tilde{t}}{2}\right) = 0. \quad (13.77)$$

Plug the Kummer functions Eq.(13.74) into the above recurrence relation, we obtain the recurrence relation among BPST vertex operators at general mass level N

$$(V^P)^{q_1} (V^T)^{N-q_1} (X)^{r_1} \left[X^2 + \left(\frac{\tilde{t}}{2} + \frac{t}{2} - q_1 - r_1\right)X + \frac{\tilde{t}}{2} \left(\frac{t}{2} + 1 - r_1\right)\right] = 0 \quad (13.78)$$

where we have defined

$$X \equiv \frac{e^P \cdot \partial X(1)}{e^L \cdot \partial X(1)} \left(\sqrt{-t}M \frac{V^L}{V^T} - \frac{\tilde{t}'}{2}\right) = \frac{e^P \cdot \partial X(1)}{e^L \cdot \partial X(1)} \left(\sqrt{-t}M \frac{V^L}{V^T} - \frac{t + M^2 + 2}{2}\right). \quad (13.79)$$

As an example, at the mass level $M^2 = 4$ with $q_1 = r_1 = 0$, we get

$$(V^T)^3 \left[X^2 + (t - 1)X + \left(\frac{t^2}{4} - 1\right)\right] = 0 \quad (13.80)$$

where

$$X = \frac{e^P \cdot \partial X(1)}{e^L \cdot \partial X(1)} \left(\sqrt{-t}M \frac{V^L}{V^T} - \frac{t + 6}{2}\right). \quad (13.81)$$

A simple calculation shows that Eq.(13.80) is exactly the same as Eq.(13.67), and the same recurrence relation among Regge string scattering amplitudes Eq.(13.68) follows.

XIV. REGGE STRING SCATTERED FROM D-PARTICLE

In this chapter we study [42] scattering of higher spin closed string states at arbitrary mass levels from D-particle in the RR. The scattering of massless string states from D-brane was well studied in the literature and can be found in [52, 135–139] Since the mass of D-brane scales as the inverse of the string coupling constant $1/g$, it was assumed that it was infinitely heavy to leading order in g and did not recoil.

We will extract the *complete* infinite ratios in Eq.(5.60) among high energy amplitudes of different string states in the fixed angle regime from these Regge string scattering amplitudes. The complete ratios calculated by this indirect method include a subset of ratios in Eq.(9.55) calculated previously by direct fixed angle calculation [41].

More importantly, we discover that the RR amplitudes calculated in this chapter for closed string D-particle scatterings can NOT be factorized and thus are different from amplitudes for the high-energy closed string-string scattering calculated previously [35, 147]. GR Amplitudes for the high-energy closed string-string scattering calculated in chapter VII can be factorized into two open string scattering amplitudes by using a calculation [147] based on the KLT formula [43]. Similarly the RR closed string-string amplitudes [35] can be factorized too. Presumably, this non-factorization is due to the non-existence of a KLT-like formula for the string D-brane scattering amplitudes. There is no physical picture for open string D-particle tree scattering amplitudes and thus no factorization for closed string D-particle scatterings into two channels of open string D-particle scatterings.

However, surprisingly, we will find [42] that in spite of the non-factorizability of the closed string D-particle scattering amplitudes, the complete ratios derived for the fixed angle regime are found to be *factorized*. These ratios are consistent with the decoupling of high-energy ZNS calculated in Eq.(5.60) of chapter V. [26–31, 33, 44].

A. Kinematics Set-up

In this chapter, we consider an incoming string state with momentum k_2 scattered from an infinitely heavy D-particle and end up with string state with momentum k_1 in the RR. The high energy scattering plane will be assumed to be the $X - Y$ plane, and the momenta

are arranged to be

$$k_1 = (E, k_1 \cos \phi, -k_1 \sin \phi), \quad (14.1)$$

$$k_2 = (-E, -k_2, 0) \quad (14.2)$$

where

$$E = \sqrt{k_2^2 + M_2^2} = \sqrt{k_1^2 + M_1^2}, \quad (14.3)$$

and ϕ is the scattering angle. For simplicity, we will calculate the disk amplitude in this paper. The relevant propagators for the left-moving string coordinate $X^\mu(z)$ and the right-moving one $\tilde{X}^\nu(\bar{w})$ are

$$\langle X^\mu(z), X^\nu(w) \rangle = -\eta^{\mu\nu} \langle X(z), X(w) \rangle = -\eta^{\mu\nu} \ln(z - w), \quad (14.4)$$

$$\langle \tilde{X}^\mu(\bar{z}), \tilde{X}^\nu(\bar{w}) \rangle = -\eta^{\mu\nu} \langle \tilde{X}(\bar{z}), \tilde{X}(\bar{w}) \rangle = -\eta^{\mu\nu} \ln(\bar{z} - \bar{w}), \quad (14.5)$$

$$\langle X^\mu(z), \tilde{X}^\nu(\bar{w}) \rangle = -D^{\mu\nu} \langle X(z), \tilde{X}(\bar{w}) \rangle = -D^{\mu\nu} \ln(1 - z\bar{w}) \quad (\text{for Disk}) \quad (14.6)$$

where matrix D has the standard form for the fields satisfying Neumann boundary condition, while D reverses the sign for the fields satisfying Dirichlet boundary condition. Instead of the Mandelstam variables used in the string-string scatterings, we define

$$a_0 \equiv k_1 \cdot D \cdot k_1 = -E^2 - k_1^2 \sim -2E^2, \quad (14.7)$$

$$a'_0 \equiv k_2 \cdot D \cdot k_2 = -E^2 - k_2^2 \sim -2E^2, \quad (14.8)$$

$$b_0 \equiv 2k_1 \cdot k_2 + 1 = 2(E^2 - k_1 k_2 \cos \phi) + 1 = \text{fixed}, \quad (14.9)$$

$$c_0 \equiv 2k_1 \cdot D \cdot k_2 + 1 = 2(E^2 + k_1 k_2 \cos \phi) + 1, \quad (14.10)$$

so that

$$2a_0 + b_0 + c_0 = 2M_1^2 + 2. \quad (14.11)$$

Since we are going to calculate Regge scattering amplitudes, $b_0 = \text{fixed}$. We can use Eq.(14.3) and Eq.(14.9) to calculate

$$\cos \phi \sim 1 - \frac{b_0 - M_1^2 - M_2^2 - 1}{2k_1^2} \quad (14.12)$$

$$\sin \phi \sim \frac{\sqrt{b_0 - M_1^2 - M_2^2 - 1}}{k_1} \equiv \frac{\sqrt{b_0}}{k_1} \quad (14.13)$$

The normalized polarization vectors on the high energy scattering plane of the k_2 string state are defined to be [26–28]

$$e_P = \frac{1}{M_2}(-E, -k_2, 0) = \frac{k_2}{M_2}, \quad (14.14)$$

$$e_L = \frac{1}{M_2}(-k_2, -E, 0), \quad (14.15)$$

$$e_T = (0, 0, 1). \quad (14.16)$$

One can then easily calculate the following kinematics

$$\begin{aligned} e^T \cdot k_2 &= 0, \\ e^T \cdot k_1 &= -k_1 \sin \phi \sim -\sqrt{\tilde{b}_0}, \\ e^T \cdot D \cdot k_1 &= k_1 \sin \phi \sim \sqrt{\tilde{b}_0}, \\ e^T \cdot D \cdot k_2 &= 0, \\ e^P \cdot k_2 &= -M_2, \\ e^P \cdot k_1 &= \frac{1}{M_2} [E^2 - k_1 k_2 \cos \phi] = \frac{b_0 - 1}{2M_2}, \\ e^P \cdot D \cdot k_1 &= \frac{1}{M_2} [E^2 + k_1 k_2 \cos \phi] = \frac{c_0 - 1}{2M_2}, \\ e^P \cdot D \cdot k_2 &= \frac{1}{M_2} [-E^2 - k_2^2] = \frac{a'_0}{M_2} \sim \frac{a_0}{M_2}, \\ e^T \cdot D \cdot e^T &= -1, \\ e^T \cdot D \cdot e^P &= e^P \cdot D \cdot e^T = 0, \\ e^P \cdot D \cdot e^P &= \frac{1}{M_2^2} [-E^2 - k_2^2] = \frac{a'_0}{M_2^2} \sim \frac{a_0}{M_2^2}, \end{aligned} \quad (14.17)$$

which will be useful in the amplitude calculation in the next section.

B. Regge String D-particle scatterings

We now begin to calculate the scattering amplitudes. For simplicity, we will take k_1 to be the tachyon and k_2 to be the tensor states. One can easily argue that a class of high energy string states for k_2 in the RR are [63, 64]

$$|p_n, p'_n, q_m, q'_m\rangle = \left[\prod_{n>0} (\alpha_{-n}^T)^{p_n} \prod_{m>0} (\alpha_{-m}^P)^{q_m} \right] \left[\prod_{n>0} (\tilde{\alpha}_{-n}^T)^{p'_n} \prod_{m>0} (\tilde{\alpha}_{-m}^P)^{q'_m} \right] |0, k\rangle \quad (14.18)$$

with

$$\sum_n n (p_n - p'_n) + \sum_m m (q_m - q'_m) = 0, \quad (14.19)$$

$$\sum_n n (p_n + p'_n) + \sum_m m (q_m + q'_m) = N = \text{const} \quad (14.20)$$

where $M_2^2 = (N - 2)$.

1. An example

Before calculating the string D-particle scattering amplitudes for general cases, we take an example and illustrate the method of calculation. We consider the case

$$p_1 = p'_1 = q_1 = q'_1 = q_2 = q'_2 = 1, \quad \text{others} = 0. \quad (14.21)$$

As we will see in the next section, the string D-particle scattering amplitudes with the general states (14.18) are reduced to simple forms in the Regge limit, in which most of the ways of contracting the operators are discarded as subleading. For a fixed number of the contractions between ∂X^P and $\bar{\partial} \tilde{X}^P$, the ways of contracting the other factors are determined by the following rules.

$$\alpha_{-n}^T \quad 1 \text{ term (contraction of } ik_1 X \text{ with } \partial_n X^T) \quad (14.22)$$

$$\tilde{\alpha}_{-n}^T \quad 1 \text{ term (contraction of } ik_1 \tilde{X} \text{ with } \bar{\partial}_n \tilde{X}^T) \quad (14.23)$$

$$\alpha_{-n}^P \quad \begin{cases} (n > 1) & 1 \text{ term (contraction of } ik_1 X \text{ with } \partial_n X^P) \\ (n = 1) & 2 \text{ terms (contraction of } ik_1 X \text{ and } ik_2 X \text{ with } \partial X^P) \end{cases} \quad (14.24)$$

$$\tilde{\alpha}_{-n}^P \quad \begin{cases} (n > 1) & 1 \text{ term (contraction of } ik_1 \tilde{X} \text{ with } \bar{\partial}_n \tilde{X}^P) \\ (n = 1) & 2 \text{ terms (contraction of } ik_1 \tilde{X} \text{ and } ik_2 \tilde{X} \text{ with } \bar{\partial} \tilde{X}^P) \end{cases} \quad (14.25)$$

Therefore we take the state Eq.(14.21) as the simplest example for the purpose of this section.

We start with the procedure in [43] to treat the vertex operator corresponding to the state (14.21).

$$\begin{aligned} V &= i^6 \varepsilon_{\mu_1 \dots \mu_6} : \partial X^{\mu_1} \partial X^{\mu_2} \partial^2 X^{\mu_3} e^{ik_2 X}(z) : : \bar{\partial} \tilde{X}^{\mu_4} \bar{\partial} \tilde{X}^{\mu_5} \bar{\partial}^2 \tilde{X}^{\mu_6} e^{ik_2 \tilde{X}}(\bar{z}) : \\ &= i^6 : \partial X^T \partial X^P \partial^2 X^P e^{ik_2 X}(z) : : \bar{\partial} \tilde{X}^T \bar{\partial} \tilde{X}^P \bar{\partial}^2 \tilde{X}^P e^{ik_2 \tilde{X}}(\bar{z}) : \\ &= i^6 \left[: \exp \left\{ ik_2 X(z) + \varepsilon_T^{(1)} \partial X^T(z) + \varepsilon_P^{(1)} \partial X^P(z) + \varepsilon_P^{(2)} \partial^2 X^P(z) \right\} : \right. \\ &\quad \left. \times : \exp \left\{ ik_2 \tilde{X}(\bar{z}) + \varepsilon_T'^{(1)} \bar{\partial} \tilde{X}^T(\bar{z}) + \varepsilon_P'^{(1)} \bar{\partial} \tilde{X}^P(\bar{z}) + \varepsilon_P'^{(2)} \bar{\partial}^2 \tilde{X}^P(\bar{z}) \right\} : \right]_{\text{linear terms}} \quad (14.26) \end{aligned}$$

In the last equation, we have introduced the dummy variables $\varepsilon_T^{(1)}, \varepsilon_P^{(1)}, \varepsilon_P^{(2)}, \varepsilon_T'^{(1)}, \varepsilon_P'^{(1)}, \varepsilon_P'^{(2)}$ associated with the non-vanishing component ε_{TPPTPP} of the polarization tensor and written

the operator in the exponential form. “linear terms” indicate that we take the sum of the terms linear in all of $\varepsilon_T^{(1)}, \varepsilon_P^{(1)}, \varepsilon_P^{(2)}, \varepsilon_T'^{(1)}, \varepsilon_P'^{(1)}$, and $\varepsilon_P'^{(2)}$. This sum can be rephrased as the coefficient of the product $\varepsilon_T^{(1)} \varepsilon_P^{(1)} \varepsilon_P^{(2)} \varepsilon_T'^{(1)} \varepsilon_P'^{(1)} \varepsilon_P'^{(2)}$ because we set the dummy variables to be 1 at the end of calculation.

The string D-particle scattering amplitudes can be calculated to be

$$\begin{aligned}
A = & \int d^2 z_1 d^2 z_2 (1 - z_1 \bar{z}_1)^{a_0} (1 - z_2 \bar{z}_2)^{a'_0} |z_1 - z_2|^{b_0-1} |1 - z_1 \bar{z}_2|^{c_0-1} \\
& \cdot \left[\exp \left\{ \begin{aligned}
& \varepsilon_T^{(1)} \left[\frac{ie^T k_1}{(z_1 - z_2)} + \frac{ie^T Dk_1 \bar{z}_1}{(1 - \bar{z}_1 z_2)} + \frac{ie^T Dk_2 \bar{z}_2}{(1 - \bar{z}_2 z_2)} \right] + \varepsilon_T'^{(1)} \left[\frac{ie^T Dk_1 z_1}{(1 - z_1 \bar{z}_2)} + \frac{ie^T k_1}{(\bar{z}_1 - \bar{z}_2)} + \frac{ie^T Dk_2 z_2}{(1 - z_2 \bar{z}_2)} \right] \\
& + \varepsilon_P^{(1)} \left[\frac{ie^P k_1}{(z_1 - z_2)} + \frac{ie^P Dk_1 \bar{z}_1}{(1 - \bar{z}_1 z_2)} + \frac{ie^P Dk_2 \bar{z}_2}{(1 - \bar{z}_2 z_2)} \right] + \varepsilon_P^{(2)} \left[\frac{ie^P k_1}{(z_1 - z_2)^2} + \frac{ie^P Dk_1 \bar{z}_1^2}{(1 - \bar{z}_1 z_2)^2} + \frac{ie^P Dk_2 \bar{z}_2^2}{(1 - \bar{z}_2 z_2)^2} \right] \\
& + \varepsilon_P'^{(1)} \left[\frac{ie^P Dk_1 z_1}{(1 - z_1 \bar{z}_2)} + \frac{ie^P k_1}{(\bar{z}_1 - \bar{z}_2)} + \frac{ie^P Dk_2 z_2}{(1 - z_2 \bar{z}_2)} \right] + \varepsilon_P'^{(2)} \left[\frac{ie^P Dk_1 z_1^2}{(1 - z_1 \bar{z}_2)^2} + \frac{ie^P k_1}{(\bar{z}_1 - \bar{z}_2)^2} + \frac{ie^P Dk_2 z_2^2}{(1 - z_2 \bar{z}_2)^2} \right] \\
& + \varepsilon_T^{(1)} \varepsilon_T'^{(1)} \frac{e^T D e^T}{(1 - z_2 \bar{z}_2)^2} \\
& + \varepsilon_P^{(1)} \varepsilon_P'^{(1)} \frac{e^P D e^P}{(1 - z_2 \bar{z}_2)^2} + 2\varepsilon_P^{(1)} \varepsilon_P'^{(2)} \frac{e^P D e^P z_2}{(1 - z_2 \bar{z}_2)^3} + 2\varepsilon_P^{(2)} \varepsilon_P'^{(1)} \frac{e^P D e^P \bar{z}_2}{(1 - z_2 \bar{z}_2)^3} + 2\varepsilon_P^{(2)} \varepsilon_P'^{(2)} \frac{e^P D e^P (1 + 2z_2 \bar{z}_2)}{(1 - z_2 \bar{z}_2)^4} \\
& + \varepsilon_T^{(1)} \varepsilon_P'^{(1)} \frac{e^T D e^P}{(1 - z_2 \bar{z}_2)^2} + 2\varepsilon_T^{(1)} \varepsilon_P'^{(2)} \frac{e^T D e^P z_2}{(1 - z_2 \bar{z}_2)^3} + \varepsilon_P^{(1)} \varepsilon_T'^{(1)} \frac{e^P D e^T}{(1 - z_2 \bar{z}_2)^2} + 2\varepsilon_P^{(2)} \varepsilon_T'^{(1)} \frac{e^P D e^T \bar{z}_2}{(1 - z_2 \bar{z}_2)^3}
\end{aligned} \right\} \Bigg]_{\text{linear terms}} \quad (14.27)
\end{aligned}$$

To fix the $SL(2, R)$ modulus group on the disk, we set $z_1 = 0$ and $z_2 = r$, then $d^2 z_1 d^2 z_2 =$

$d(r^2)$. By using Eq.(14.17), the amplitude can then be reduced to

$$\begin{aligned}
A = & \int_0^1 d(r^2) (1-r^2)^{a'_0} r^{b_0-1} \\
& \cdot \left[\exp \left\{ \begin{aligned}
& \varepsilon_T^{(1)} \left[-\frac{i\sqrt{\tilde{b}_0}}{-r} \right] + \varepsilon_T^{\prime(1)} \left[-\frac{i\sqrt{\tilde{b}_0}}{-r} \right] \\
& + \varepsilon_P^{(1)} \left[\frac{i\frac{b_0-1}{2M_2}}{-r} + \frac{i\frac{a_0}{M_2}}{(1-r^2)/r} \right] + \varepsilon_P^{(2)} \left[\frac{i\frac{b_0-1}{2M_2}}{(-r)^2} + \frac{i\frac{a_0}{M_2}}{[(1-r^2)/r]^2} \right] \\
& + \varepsilon_P^{\prime(1)} \left[\frac{i\frac{b_0-1}{2M_2}}{-r} + \frac{i\frac{a_0}{M_2}}{(1-r^2)/r} \right] + \varepsilon_P^{\prime(2)} \left[\frac{i\frac{b_0-1}{2M_2}}{(-r)^2} + \frac{i\frac{a_0}{M_2}}{[1-r^2/r]^2} \right] \\
& - \varepsilon_T^{(1)} \varepsilon_T^{\prime(1)} \frac{1}{(1-r^2)^2} \\
& + \varepsilon_P^{(1)} \varepsilon_P^{\prime(1)} \frac{\frac{a_0}{M_2^2}}{(1-r^2)^2} + 2\varepsilon_P^{(1)} \varepsilon_P^{\prime(2)} \frac{\frac{a_0}{M_2^2} r}{(1-r^2)^3} + 2\varepsilon_P^{(2)} \varepsilon_P^{\prime(1)} \frac{\frac{a_0}{M_2^2} r}{(1-r^2)^3} + 2\varepsilon_P^{(2)} \varepsilon_P^{\prime(2)} \frac{\frac{a_0}{M_2^2} (1+2r^2)}{(1-r^2)^4}
\end{aligned} \right\} \right]_{\text{linear terms}}
\end{aligned} \tag{14.28}$$

Although in Eq.(14.28) we have dropped several subleading terms by using the kinematic relations Eq.(14.17), Eq.(14.28) still has subleading terms. We can see that by performing the integration of a generic term in Eq.(14.28) and looking at its behavior in the Regge limit explicitly.

$$\begin{aligned}
\int_0^1 d(r^2) (1-r^2)^{a'_0+n_a} r^{b_0-1-N+n_b} &= B \left(a'_0 + 1 + n_a, \frac{b_0 - N + 1}{2} + \frac{n_b}{2} \right) \\
&= B \left(a'_0 + 1, \frac{b_0 - N + 1}{2} \right) \frac{(a'_0 + 1)_{n_a} \left(\frac{b_0 - N + 1}{2} \right)_{\frac{n_b}{2}}}{(a'_0 + 1 + \frac{b_0 - N + 1}{2})_{n_a + \frac{n_b}{2}}} \\
&\sim B \left(a_0 + 1, \frac{b_0 - N + 1}{2} \right) \left(\frac{b_0 - N + 1}{2} \right)_{\frac{n_b}{2}} (a_0)^{-\frac{n_b}{2}}
\end{aligned} \tag{14.29}$$

Here the Pochhammer symbol is defined by $(x)_y = \frac{\Gamma(x+y)}{\Gamma(x)}$, which, if y is a positive integer, is reduced to $(x)_y = x(x+1)(x+2) \cdots (x+y-1)$. From the Regge behavior Eq.(14.29), we see that increasing one power of $1/r$ in the integrand results in increasing one-half power of a_0 . Thus we obtain the following rules to determine which terms in the exponent of Eq.(14.28) contribute to the leading behavior of the amplitude:

$$1/r \rightarrow E, \quad a_0 \rightarrow E^2. \tag{14.30}$$

We can now drop the subleading terms in energy to get

$$\begin{aligned}
A = & \int_0^1 d(r^2) (1-r^2)^{a'_0} r^{b_0-1} \\
& \cdot \left[\exp \left\{ \varepsilon_T^{(1)} \left[-\frac{i\sqrt{\tilde{b}_0}}{-r} \right] + \varepsilon_T^{\prime(1)} \left[-\frac{i\sqrt{\tilde{b}_0}}{-r} \right] + \varepsilon_P^{(2)} \left[\frac{i\frac{b_0-1}{2M_2}}{(-r)^2} \right] + \varepsilon_P^{\prime(2)} \left[\frac{i\frac{b_0-1}{2M_2}}{(-r)^2} \right] \right\} \right]_{\epsilon_{TPTP}} \\
& \cdot \left[\exp \left\{ \varepsilon_P^{(1)} \left[\frac{i\frac{b_0-1}{2M_2}}{-r} + \frac{i\frac{a_0}{M_2}}{(1-r^2)/r} \right] + \varepsilon_P^{\prime(1)} \left[\frac{i\frac{b_0-1}{2M_2}}{-r} + \frac{i\frac{a_0}{M_2}}{(1-r^2)/r} \right] + \varepsilon_P^{(1)} \varepsilon_P^{\prime(1)} \frac{\frac{a_0}{M_2^2}}{(1-r^2)^2} \right\} \right]_{\epsilon_{PP}}
\end{aligned} \tag{14.31}$$

where $[\dots]_{\epsilon_{TPTP}}$ in the second line and $[\dots]_{\epsilon_{PP}}$ in the third line indicate that we take the coefficients of $\varepsilon_T^{(1)} \varepsilon_T^{\prime(1)} \varepsilon_P^{(2)} \varepsilon_P^{\prime(2)}$ and $\varepsilon_P^{(1)} \varepsilon_P^{\prime(1)}$ respectively. Because of the difference in the powers of $1/r$ and a_0 in the exponent of Eq.(14.28), Eq.(14.31) has much more structure for $\varepsilon_P^{(1)}$ and $\varepsilon_P^{\prime(1)}$ than for $\varepsilon_T^{(1)}$, $\varepsilon_T^{\prime(1)}$, $\varepsilon_P^{(2)}$, and $\varepsilon_P^{\prime(2)}$, and fits into the rules Eqs.(14.22),(14.23),(14.24) and (14.25). It is also worth noting that the appearance of the last term in the second exponent of Eq.(14.31) originates from the contraction between $\partial X(z_2)$ and $\bar{\partial} \tilde{X}(\bar{z}_2)$ in Eq.(14.27), which is a characteristic of string D-brane scattering.

The explicit form of the amplitude for the current example is

$$\begin{aligned}
A = & \int_0^1 d(r^2) (1-r^2)^{a'_0} r^{b_0-1} \left(-\frac{i\sqrt{\tilde{b}_0}}{-r} \right) \left(-\frac{i\sqrt{\tilde{b}_0}}{-r} \right) \left(\frac{i\frac{b_0-1}{2M_2}}{(-r)^2} \right) \left(\frac{i\frac{b_0-1}{2M_2}}{(-r)^2} \right) \\
& \cdot \left[\left(\frac{i\frac{b_0-1}{2M_2}}{-r} + \frac{i\frac{a_0}{M_2}}{(1-r^2)/r} \right) \left(\frac{i\frac{b_0-1}{2M_2}}{-r} + \frac{i\frac{a_0}{M_2}}{(1-r^2)/r} \right) + \frac{\frac{a_0}{M_2^2}}{(1-r^2)^2} \right]
\end{aligned} \tag{14.32}$$

$$\begin{aligned}
= & - \left(\sqrt{\tilde{b}_0} \right)^2 \left(\frac{b_0-1}{2M_2} \right)^4 \int_0^1 d(r^2) (1-r^2)^{a'_0} r^{b_0-9} \\
& \cdot \left[\left(\sum_{l=0}^2 \binom{2}{l} \left(\frac{-r^2}{(1-r^2)} \frac{2a_0}{b_0-1} \right)^l \right) - \frac{r^2}{(1-r^2)^2} \frac{4a_0}{(b_0-1)^2} \right]
\end{aligned} \tag{14.33}$$

$$\begin{aligned}
\sim & - \left(\sqrt{\tilde{b}_0} \right)^2 \left(\frac{b_0-1}{2M_2} \right)^4 B \left(a_0+1, \frac{b_0-7}{2} \right) \\
& \cdot \left[\left(\sum_{l=0}^2 \binom{2}{l} \left(-\frac{2}{b_0-1} \right)^l \left(\frac{b_0-7}{2} \right)_l \right) - \frac{4}{(b_0-1)^2} \left(\frac{b_0-7}{2} \right) \right]
\end{aligned} \tag{14.34}$$

$$\begin{aligned}
= & - \left(\sqrt{\tilde{b}_0} \right)^2 \left(\frac{b_0-1}{2M_2} \right)^4 B \left(a_0+1, \frac{b_0-7}{2} \right) \\
& \cdot \left[{}_2F_0 \left(-2, \frac{b_0-7}{2}, \frac{2}{b_0-1} \right) - \frac{4}{(b_0-1)^2} \left(\frac{b_0-7}{2} \right) \right]
\end{aligned} \tag{14.35}$$

where we have used Eq.(14.29).

2. General cases

Now we move on to general cases. The vertex operator corresponding to a general massive state with d left-modes and d' right-modes is of the following form.

$$V = i^{d+d'} \varepsilon_{\mu_1 \dots \mu_{d+d'}} : \partial^{n_1} X^{\mu_1} \dots \partial^{n_d} X^{\mu_d} e^{ik_2 X}(z) : : \bar{\partial}^{n_{d+1}} \tilde{X}^{\mu_{d+1}} \dots \bar{\partial}^{n_{d+d'}} \tilde{X}^{\mu_{d+d'}} e^{ik_2 \tilde{X}}(\bar{z}) : \quad (14.36)$$

The vertex operators corresponding to the states Eq.(14.18) are expressed in this covariant form by

$$d = \sum_{n>0} p_n + q_n, \quad d' = \sum_{n>0} p'_n + q'_n$$

$$(n_1, n_2, \dots, n_{d+d'}) = \left(\dots, \underbrace{m, \dots, m}_{p_m}, \dots, \underbrace{n, \dots, n}_{q_n}, \dots, \underbrace{m', \dots, m'}_{p'_{m'}}, \dots, \underbrace{n', \dots, n'}_{q'_{n'}}, \dots \right)$$

$$\varepsilon \dots \underbrace{T \dots T}_{p_m} \dots \underbrace{P \dots P}_{q_n} \dots \underbrace{T \dots T}_{p'_{m'}} \dots \underbrace{P \dots P}_{q'_{n'}} = 1.$$

For the calculation of the correlator involving the operator Eq.(14.36), we introduce parameters associated with the polarization tensor and exponentiate the kinematic factors.

$$\varepsilon_{TTT\dots PPP\dots TTT\dots PPP\dots} \rightarrow \prod_{n>0} \prod_{i=1}^{p_n} \prod_{j=1}^{q_n} \prod_{i'=1}^{p'_n} \prod_{j'=1}^{q'_n} \varepsilon_{T_i}^{(n)} \varepsilon_{P_j}^{(n)} \varepsilon_{T_{i'}}^{\prime(n)} \varepsilon_{P_{j'}}^{\prime(n)}$$

$$V = (i)^{\sum_{n>0} p_n + p'_n + q_n + q'_n} \left[: \exp \left\{ ik_2 X(z) + \sum_{n>0} \sum_{i=1}^{p_n} \varepsilon_{T_i}^{(n)} \partial^n X^T(z) + \sum_{m>0} \sum_{j=1}^{q_m} \varepsilon_{P_j}^{(m)} \partial^m X^P(z) \right\} : \right.$$

$$\times : \exp \left\{ ik_2 \tilde{X}(\bar{z}) + \sum_{n>0} \sum_{i=1}^{p'_n} \varepsilon_{T_i}^{\prime(n)} \partial^n \tilde{X}^T(\bar{z}) + \sum_{m>0} \sum_{j=1}^{q'_m} \varepsilon_{P_j}^{\prime(m)} \partial^m \tilde{X}^P(\bar{z}) \right\} : \left. \right]_{\text{linear terms}} \quad (14.37)$$

where “linear terms” means the terms linear in all of $\varepsilon_{T_i}^{(n)}, \varepsilon_{P_j}^{(m)}, \varepsilon_{T_i}^{\prime(n)}$, and $\varepsilon_{P_j}^{\prime(m)}$. Below we use symbols like

$$\varepsilon_{T^3 P^2 T P^3} \equiv \varepsilon_{T_1}^{(1)} \varepsilon_{T_1}^{(3)} \varepsilon_{T_2}^{(3)} \varepsilon_{P_1}^{(2)} \varepsilon_{P_1}^{(5)} \varepsilon_{T_1}^{\prime(1)} \varepsilon_{P_1}^{\prime(1)} \varepsilon_{P_2}^{\prime(1)} \varepsilon_{P_1}^{\prime(2)}, \quad \varepsilon_T \sum_n p_n \equiv \sum_{n>0} \sum_{i=1}^{p_n} \varepsilon_{T_i}^{(n)}$$

(the meanings of these symbols are not unique.) and do not write the normal ordering symbol $: :$ to avoid messy expressions.

The string D-particle scattering amplitudes of these string states can be calculated to be

$$A = \int d^2 z_1 d^2 z_2 \cdot \varepsilon_{T \Sigma p_n P \Sigma q_n T \Sigma p'_n P \Sigma q'_n} \quad (14.38)$$

$$\cdot \left\langle e^{ik_1 X}(z_1) e^{ik_1 \tilde{X}}(\bar{z}_1) \cdot \prod_{n>0} (i\partial^n X^T)^{p_n} \prod_{m>0} (i\partial^m X^P)^{q_m} e^{ik_2 X}(z_2) \right. \\ \left. \cdot \prod_{n>0} (i\bar{\partial}^n \tilde{X}^T)^{p'_n} \prod_{m>0} (i\bar{\partial}^m \tilde{X}^P)^{q'_m} e^{ik_2 \tilde{X}}(\bar{z}_2) \right\rangle \\ \equiv (i)^{\sum_{n>0} p_n + p'_n + q_n + q'_n} A' \quad (14.39)$$

$$= (i)^{\sum_{n>0} p_n + p'_n + q_n + q'_n} \int d^2 z_1 d^2 z_2 \\ \cdot \exp \left\{ \begin{aligned} & \left\langle (ik_1 X)(z_1) (ik_1 \tilde{X})(\bar{z}_1) \right\rangle \\ & + \left\langle \left(\varepsilon_T \sum_{n>0} p_n \partial^n X^T + \varepsilon_P \sum_{m>0} q_m \partial^m X^P + ik_2 X \right)(z_2) \right\rangle \\ & + \left\langle \left(\varepsilon'_T \sum_{n>0} p'_n \bar{\partial}^n \tilde{X}^T + \varepsilon'_P \sum_{m>0} q'_m \bar{\partial}^m \tilde{X}^P + ik_2 \tilde{X} \right)(\bar{z}_2) \right\rangle \\ & + \left\langle (ik_1 X)(z_1) \left(\varepsilon_T \sum_{n>0} p_n \partial^n X^T + \varepsilon_P \sum_{m>0} q_m \partial^m X^P + ik_2 X \right)(z_2) \right\rangle \\ & + \left\langle (ik_1 \tilde{X})(\bar{z}_1) \left(\varepsilon'_T \sum_{n>0} p'_n \bar{\partial}^n \tilde{X}^T + \varepsilon'_P \sum_{m>0} q'_m \bar{\partial}^m \tilde{X}^P + ik_2 \tilde{X} \right)(\bar{z}_2) \right\rangle \\ & + \left\langle (ik_1 X)(z_1) \left(\varepsilon'_T \sum_{n>0} p'_n \bar{\partial}^n \tilde{X}^T + \varepsilon'_P \sum_{m>0} q'_m \bar{\partial}^m \tilde{X}^P + ik_2 \tilde{X} \right)(\bar{z}_2) \right\rangle \\ & + \left\langle (ik_1 \tilde{X})(\bar{z}_1) \left(\varepsilon_T \sum_{n>0} p_n \partial^n X^T + \varepsilon_P \sum_{m>0} q_m \partial^m X^P + ik_2 X \right)(z_2) \right\rangle \end{aligned} \right\} \quad (14.40)$$

where only linear terms are taken in the expansion of the exponential (in the sense of Eq.(14.37)). In Eq.(14.40), we have used the simplified notation $\varepsilon_{T_j}^{(n)} \equiv \varepsilon_T$, $j = 1, 2, \dots, p_n$, $n \in Z_+$ for the spin polarizations, and similarly for the other polarizations. Note that there will be terms corresponding to quadratic in the spin polarization. After fixing the $SL(2, R)$ modulus group on the disk, we set $z_1 = 0$ and $z_2 = r$, then $d^2 z_1 d^2 z_2 = d(r^2)$. By using

Eq.(14.17), the amplitude can then be reduced to

$$A' = \int d(r^2) (1 - r^2)^{a'_0} r^{b_0-1} \exp \left\{ \begin{aligned} & \varepsilon_T \sum_{n>0} p_n \left[-\frac{i(n-1)! \sqrt{b_0}}{(-r)^n} \right] + \varepsilon'_T \sum_{n'>0} p'_{n'} \left[-\frac{i(n'-1)! \sqrt{b_0}}{(-r)^{n'}} \right] \\ & + \varepsilon_P \sum_{m>0} q_m \left[\frac{i(m-1)! \frac{b_0-1}{2M_2}}{(-r)^m} + \frac{i(m-1)! \frac{a_0}{M_2}}{[(1-r^2)/r]^m} \right] \\ & + \varepsilon'_P \sum_{m'>0} q'_{m'} \left[\frac{i(m'-1)! \frac{b_0-1}{2M_2}}{(-r)^{m'}} + \frac{i(m'-1)! \frac{a_0}{M_2}}{[(1-r^2)/r]^{m'}} \right] \\ & - \varepsilon_T \varepsilon'_T \sum_{n,n'>0} p_n p'_{n'} \partial^n \bar{\partial}^{n'} \ln(1 - z_2 \bar{z}_2) \Big|_{z_2=\bar{z}_2=r} \\ & - \varepsilon_P \varepsilon'_P \sum_{m,m'>0} q_m q'_{m'} \partial^m \bar{\partial}^{m'} \ln(1 - z_2 \bar{z}_2) \Big|_{z_2=\bar{z}_2=r} \frac{a_0}{M_2^2} \end{aligned} \right\} \quad (14.41)$$

where only linear terms are taken in the expansion of the exponential.

Now we use the energy counting Eq.(14.30) and show how we reach the rules Eqs.(14.22),(14.23),(14.24) and (14.25). We can see immediately that in the exponent of Eq.(14.41), the terms linear in $\varepsilon_{P_i}^{(n)}$ or $\varepsilon'_{P_i}{}^{(n)}$ are dominated by their first terms if $m \geq 2$ or $m' \geq 2$. We can see also that most of the terms in the forth and fifth lines of the exponent are discarded as subleading. If we start with the terms consisting of only the factors coming from the first three lines, the other terms are obtained by series of replacements of two factors in them with one factors coming from the forth and fifth lines, and for each of the replacements we can see how it changes the power of energy. We do not need to calculate the infinite number of derivatives. For each differentiation the increase of the power of $1/r$ is less than or equal to 1, while the powers of $1/r$ in the first three lines increase with n, n', m or m' , which implies that if one term in the forth or fifth line is discarded, the terms with higher n, n', m, m' in the same line are also discarded. The sequences of those discarded terms start at $(n, n') = (1, 1)$, $(m, m') = (1, 2)$, and $(m, m') = (2, 1)$. In this way, we can see that only the terms with $m = m' = 1$ in the fifth line contribute to the leading behavior.

Thus we obtain the generalization of Eq.(14.31)

$$\begin{aligned}
A' = & \int d(r^2) (1-r^2)^{a'_0} r^{b_0-1} \\
& \exp \left\{ \varepsilon_T \sum_{n>0} p_n \left[-\frac{i(n-1)!\sqrt{\tilde{b}_0}}{(-r)^n} \right] + \varepsilon'_T \sum_{n'>0} p'_{n'} \left[-\frac{i(n'-1)!\sqrt{\tilde{b}_0}}{(-r)^{n'}} \right] \right. \\
& \left. + \varepsilon_P \sum_{m>1} q_m \left[\frac{i(m-1)! \frac{b_0-1}{2M_2}}{(-r)^m} \right] + \varepsilon'_P \sum_{m'>1} q'_{m'} \left[\frac{i(m'-1)! \frac{b_0-1}{2M_2}}{(-r)^{m'}} \right] \right\}^{\varepsilon_T \sum p_n P \sum' q_n T \sum p'_{n'} P \sum' q'_{n'}} \\
& \exp \left\{ \varepsilon_P q_1 \left[\frac{i \frac{b_0-1}{2M_2}}{-r} + \frac{i \frac{a_0}{M_2} r}{1-r^2} \right] + \varepsilon'_P q'_1 \left[\frac{i \frac{b_0-1}{2M_2}}{-r} + \frac{i \frac{a_0}{M_2} r}{1-r^2} \right] + \varepsilon_P \varepsilon'_P q_1 q'_1 \frac{\frac{a_0}{M_2^2}}{(1-r^2)^2} \right\}^{\varepsilon_{Pq_1 Pq'_1}}
\end{aligned} \tag{14.42}$$

where the symbols $\varepsilon \dots$ are similar to the ones in Eq.(14.31) and indicate that we take the coefficients of the products of the dummy variables in the exponents. ($\varepsilon_{P_i}^{(1)}$ and $\varepsilon_{P_i}'^{(1)}$ are excluded in the “sums” \sum' .) Note that the last term in the last line of Eq.(14.42) is quadratic in the polarization. This term is a characteristic of string D-brane scattering and has no analog in any of the previous works. It will play a crucial role in the following calculation in this paper.

For further calculation, we first note that

$$\begin{aligned}
& \exp \left\{ \varepsilon_P q_1 \left[\frac{i \frac{b_0-1}{2M_2}}{-r} + \frac{i \frac{a_0}{M_2} r}{1-r^2} \right] + \varepsilon'_P q'_1 \left[\frac{i \frac{b_0-1}{2M_2}}{-r} + \frac{i \frac{a_0}{M_2} r}{1-r^2} \right] + \varepsilon_P \varepsilon'_P q_1 q'_1 \frac{\frac{a_0}{M_2^2}}{(1-r^2)^2} \right\}^{\varepsilon_{Pq_1 Pq'_1}} \\
& = \varepsilon_{Pq_1 Pq'_1} \sum_{j=0}^{\min\{q_1, q'_1\}} \binom{q_1}{j} \binom{q'_1}{j} j! \left(\frac{i \frac{b_0-1}{2M_2}}{-r} + \frac{i \frac{a_0}{M_2} r}{1-r^2} \right)^{q_1+q'_1-2j} \left(\frac{\frac{a_0}{M_2^2}}{(1-r^2)^2} \right)^j.
\end{aligned} \tag{14.43}$$

Thus the amplitude can be further reduced to

$$\begin{aligned}
A' = & \int d(r^2) (1-r^2)^{a'_0} r^{b_0-1} \\
& \cdot \prod_{n>0} \left[-\frac{i(n-1)!\sqrt{\tilde{b}_0}}{(-r)^n} \right]^{p_n} \prod_{n'>0} \left[-\frac{i(n'-1)!\sqrt{\tilde{b}_0}}{(-r)^{n'}} \right]^{p'_{n'}} \\
& \cdot \prod_{m>1} \left[\frac{i(m-1)! \frac{b_0-1}{2M_2}}{(-r)^m} \right]^{q_m} \prod_{m'>1} \left[\frac{i(m'-1)! \frac{b_0-1}{2M_2}}{(-r)^{m'}} \right]^{q_{m'}} \\
& \cdot \sum_{j=0}^{\min\{q_1, q'_1\}} \sum_{l=0}^{q_1+q'_1-2j} j! \binom{q_1}{j} \binom{q'_1}{j} \binom{q_1+q'_1-2j}{l} \\
& \cdot \left(\frac{i \frac{b_0-1}{2M_2}}{-r} \right)^{q_1+q'_1-2j-l} \left(\frac{i \frac{a_0}{M_2} r}{1-r^2} \right)^l \left(\frac{\frac{a_0}{M_2^2}}{(1-r^2)^2} \right)^j,
\end{aligned} \tag{14.44}$$

which, in the case of the state (14.21), is reduced to Eq.(14.33). We can now do the integration to get

$$\begin{aligned}
A' &= \left(i \frac{b_0 - 1}{2M_2}\right)^{q_1 + q'_1} \cdot \prod_{n>0} \left(\left[-i(n-1)! \sqrt{\tilde{b}_0} \right]^{p_n} \left[-i(n-1)! \sqrt{\tilde{b}_0} \right]^{p'_n} \right) \\
&\cdot \prod_{m>1} \left(\left[i(m-1)! \frac{b_0 - 1}{2M_2} \right]^{q_m} \left[i(m-1)! \frac{b_0 - 1}{2M_2} \right]^{q'_m} \right) \\
&\cdot \sum_{j=0}^{\min\{q_1, q'_1\}} \sum_{l=0}^{q_1 + q'_1 - 2j} j! \binom{q_1}{j} \binom{q'_1}{j} \binom{q_1 + q'_1 - 2j}{l} \left(\frac{-2}{b_0 - 1} \right)^l \left(\frac{-4}{(b_0 - 1)^2} \right)^j \\
&\cdot B \left(a_0 + 1, \frac{b_0 + 1 - N}{2} \right) \left(\frac{b_0 + 1 - N}{2} \right)_j \left(\frac{b_0 + 1 - N}{2} + j \right)_l
\end{aligned} \tag{14.45}$$

where we have done the expansion of the beta function in the RR as following

$$\begin{aligned}
&B \left(a'_0 + 1 - l - 2j, \frac{b_0 + 1 - N}{2} + l + j \right) \\
&\approx B \left(a_0 + 1, \frac{b_0 + 1 - N}{2} \right) \frac{\left(\frac{b_0 + 1 - N}{2} \right)_{l+j}}{a_0^{l+j}} \\
&= B \left(a_0 + 1, \frac{b_0 + 1 - N}{2} \right) \frac{\left(\frac{b_0 + 1 - N}{2} \right)_j \left(\frac{b_0 + 1 - N}{2} + j \right)_l}{a_0^{l+j}}.
\end{aligned} \tag{14.46}$$

Note that in the case of the state Eq.(14.21), Eq.(14.45) is reduced to Eq.(14.34). Performing the summation over n , we obtain

$$\begin{aligned}
A' &= \left(i \frac{b_0 - 1}{2M_2}\right)^{q_1 + q'_1} \cdot \prod_{n>0} \left(\left[-i(n-1)! \sqrt{\tilde{b}_0} \right]^{p_n + p'_n} \right) \prod_{m>1} \left(\left[i(m-1)! \frac{b_0 - 1}{2M_2} \right]^{q_m + q'_m} \right) \\
&\cdot B \left(a_0 + 1, \frac{b_0 + 1 - N}{2} \right) \sum_{j=0}^{\min\{q_1, q'_1\}} (-1)^j j! \binom{q_1}{j} \binom{q'_1}{j} \left(\frac{b_0 + 1 - N}{2} \right)_j \left(\frac{2}{b_0 - 1} \right)^{2j} \\
&\cdot {}_2F_0 \left(-q_1 - q'_1 + 2j, \frac{b_0 + 1 - N}{2} + j, \frac{2}{b_0 - 1} \right),
\end{aligned} \tag{14.47}$$

which, in the case of the state Eq.(14.21), is reduced to Eq.(14.35). Finally we can use the

identity of the Kummer function

$$\begin{aligned}
& 2^{2m} \tilde{t}^{-2m} U \left(-2m, \frac{t}{2} + 2 - 2m, \frac{\tilde{t}}{2} \right) \\
&= {}_2F_0 \left(-2m, -1 - \frac{t}{2}, -\frac{2}{\tilde{t}} \right) \\
&\equiv \sum_{j=0}^{2m} (-2m)_j \left(-1 - \frac{t}{2} \right)_j \frac{\left(-\frac{2}{\tilde{t}} \right)^j}{j!} \\
&= \sum_{j=0}^{2m} \binom{2m}{j} \left(-1 - \frac{t}{2} \right)_j \left(\frac{2}{\tilde{t}} \right)^j
\end{aligned} \tag{14.48}$$

to get the final form of the amplitude

$$\begin{aligned}
A' &= \prod_{n>0} \left(\left[-i(n-1)! \sqrt{\tilde{b}_0} \right]^{p_n+p'_n} \right) \prod_{m>1} \left(\left[i(m-1)! \frac{b_0-1}{2M_2} \right]^{q_m+q'_m} \right) \left(-\frac{i}{M_2} \right)^{q_1+q'_1} \\
&\cdot B \left(a_0 + 1, \frac{b_0 + 1 - N}{2} \right) \sum_{j=0}^{\min\{q_1, q'_1\}} (-1)^j j! \binom{q_1}{j} \binom{q'_1}{j} \left(\frac{b_0 + 1 - N}{2} \right)_j \\
&\cdot U \left(-q_1 - q'_1 + 2j, \frac{-b_0 + N + 1}{2} - q_1 - q'_1 + j, -\frac{b_0 - 1}{2} \right).
\end{aligned} \tag{14.49}$$

Note that the amplitudes in Eq.(14.49) can *NOT* be factorized into two open string D-particle scattering amplitudes as in the case of closed string-string scattering amplitudes [35, 147].

An interesting application of Eq.(14.49) is the universal power law behavior of the amplitudes. We first define the Mandelstam variables as $s = 2E^2$ and $t = -(k_1 + k_2)^2$. The second argument of the beta function in Eq.(14.49) can be calculated to be

$$\frac{b_0 + 1 - N}{2} = \frac{2k_1 \cdot k_2 + 1 + 1 - N}{2} = \frac{(k_1 + k_2)^2 - k_1^2 - k_2^2 + 2 - N}{2} = \frac{-t - 2}{2} \tag{14.50}$$

where we have used Eq.(14.9) and $M_2^2 = (N - 2)$. The amplitudes thus give the universal power-law behavior for string states at *all* mass levels

$$A \sim s^{\alpha(t)} \quad (\text{in the RR}) \tag{14.51}$$

where

$$\alpha(t) = a(0) + \alpha' t, \quad a(0) = 1 \text{ and } \alpha' = \frac{1}{2}. \tag{14.52}$$

C. Reproducing ratios at the fixed angle regime

To compare the RR amplitudes Eq.(14.49) with the fixed angle amplitudes corresponding to states in Eq.(5.67), we consider the RR amplitudes of the following closed string states

$$|N; 2m, 2m'; q, q'\rangle = (\alpha_{-1}^T)^{N/2-2m-2q} (\alpha_{-1}^P)^{2m} (\alpha_{-2}^P)^q \otimes (\tilde{\alpha}_{-1}^T)^{N/2-2m'-2q'} (\tilde{\alpha}_{-1}^P)^{2m'} (\tilde{\alpha}_{-2}^P)^{q'} |0, k\rangle. \quad (14.53)$$

where m, m', q and q' are non-negative integers. We can take the following values

$$p_1 = N/2 - 2m - 2q, p'_1 = N/2 - 2m' - 2q', \quad (14.54)$$

$$q_1 = 2m, q'_1 = 2m', \quad (14.55)$$

$$q_2 = q, q'_2 = q' \quad (14.56)$$

in Eq.(14.49), and include the phase factor in Eq.(14.39) to get

$$\begin{aligned} A^{(N; 2m, 2m'; q, q')} &= (i)^{N-q-q'} \left(-i\sqrt{\tilde{b}_0} \right)^{N-2(m+m')-2(q+q')} \left(i\frac{b_0-1}{2M_2} \right)^{q+q'} \left(-\frac{i}{M_2} \right)^{2m+2m'} \\ &\cdot B \left(a_0 + 1, \frac{b_0 + 1 - N}{2} \right) \sum_{j=0}^{\min\{2m, 2m'\}} (-1)^j j! \binom{2m}{j} \binom{2m'}{j} \left(\frac{b_0 + 1 - N}{2} \right)_j \\ &\cdot U \left(-2m - 2m' + 2j, \frac{-b_0 + N + 1}{2} - 2m - 2m' + j, -\frac{b_0 - 1}{2} \right). \end{aligned} \quad (14.57)$$

It is now easy to calculate the RR ratios for each fixed mass level

$$\begin{aligned} \frac{A^{(N; 2m, 2m'; q, q')}}{A^{(N, 0, 0, 0, 0)}} &= (i)^{-q-q'} \left(-i\frac{b_0-1}{2\tilde{b}_0 M_2} \right)^{q+q'} \left(\frac{1}{\tilde{b}_0 M_2^2} \right)^{m+m'} \\ &\cdot \sum_{j=0}^{\min\{2m, 2m'\}} (-1)^j j! \binom{2m}{j} \binom{2m'}{j} \left(\frac{b_0 + 1 - N}{2} \right)_j \\ &\cdot U \left(-2m - 2m' + 2j, \frac{-b_0 + N + 1}{2} - 2m - 2m' + j, -\frac{b_0 - 1}{2} \right) \end{aligned} \quad (14.58)$$

which is a b_0 -dependent function.

Before studying the fixed angle ratios for string D-particle scatterings, we first make a pause to review previous results on *string-string* scatterings.

1. String-string scatterings

a. Open string For open string-string scatterings, either the saddle-point method ($t-u$ channel only) or the decoupling of high energy ZNS (ZNS) discussed in chapter V can be

used to calculate the fixed angle ratios [26–31, 44]. It was discovered that there was an interesting link between high energy fixed angle amplitudes T and RR amplitudes A . To the leading order in energy, the ratios among fixed angle amplitudes are ϕ -independent numbers, whereas the ratios among RR amplitudes are t -dependent functions. However, It was discovered [63] in chapter XI.B that the coefficients of the high energy RR ratios in the leading power of t can be identified with the fixed angle ratios, namely [63]

$$\lim_{\tilde{t}' \rightarrow \infty} \frac{A^{(N,2m,q)}}{A^{(N,0,0)}} = \left(-\frac{1}{M_2}\right)^{2m+q} \left(\frac{1}{2}\right)^{m+q} (2m-1)!! = \frac{T^{(N,2m,q)}}{T^{(N,0,0)}}. \quad (14.59)$$

To ensure this identification, one needs the following identity [63, 64, 66, 147]

$$\begin{aligned} & \sum_{j=0}^{2m} (-2m)_j \left(-L - \frac{\tilde{t}'}{2}\right)_j \frac{(-2/\tilde{t}')^j}{j!} \\ &= 0(-\tilde{t}')^0 + 0(-\tilde{t}')^{-1} + \dots + 0(-\tilde{t}')^{-m+1} + \frac{(2m)!}{m!} (-\tilde{t}')^{-m} + O\left\{\left(\frac{1}{\tilde{t}'}\right)^{m+1}\right\} \end{aligned} \quad (14.60)$$

where $L = 1 - N$ and is an integer. Note that L effects only the subleading terms in $O\left\{\left(\frac{1}{\tilde{t}'}\right)^{m+1}\right\}$. Mathematically, the complete proof of Eq.(14.60) for *arbitrary real values* L was worked out in [66] by using an identity of signless Stirling number of the first kind in combinatorial theory.

b. Open superstring For all four classes [33] of high energy fixed angle open superstring scattering amplitudes considered in chapter VIII, both the corresponding RR amplitudes and the complete ratios of the leading (in t) RR amplitudes considered in chapter XII can be calculated [64]. For the fixed angle regime [33], the complete ratios can be calculated by the decoupling of high energy ZNS. It turns out that the identification in Eq.(14.59) continues to work, and L is an integer again for this case [64].

c. Compactified open string For compactified open string scatterings, both the amplitudes and the complete ratios of leading (in t) RR can be calculated [67]. For the fixed angle regime or GR, the complete ratios can be calculated by the decoupling of high energy ZNS. The identification in Eq.(14.59) continues to work. However, only a subset of scattering amplitudes corresponding to the case $m = 0$ was calculated. The difficulties has been as following. First, it seems that the saddle-point method is not applicable here. On the other hand, it was shown that [26–29] the leading order amplitudes containing $(\alpha_{-1}^L)^{2m}$ component will drop from energy order E^{4m} to E^{2m} , and one needs to calculate the complicated naive subleading order terms in order to get the real leading order amplitude. One encounters this

difficulty even for some cases in the non-compactified string calculation. In these cases, the method of decoupling of high energy ZNS was adopted.

It was important to discover [67] that the identity in Eq.(14.60) for arbitrary real values L can only be realized in high energy *compactified* string scatterings. This is due to the dependence of the value L on winding momenta K_i^{25} [67]

$$L = 1 - N - (K_2^{25})^2 + K_2^{25} K_3^{25}. \quad (14.61)$$

All other high energy string scatterings calculated previously [63, 64, 147] correspond to integer value of L only.

d. Closed string For closed string scatterings [147], one can use the KLT formula [43], which expresses the relation between tree amplitudes of closed and two channels of open string ($\alpha'_{\text{closed}} = 4\alpha'_{\text{open}} = 2$), to simplify the calculations. Both ratios of leading (in t) RR amplitudes and GR amplitudes were found to be the tensor product of two ratios in Eq.(14.59), namely [147]

$$\begin{aligned} \lim_{\tilde{t}' \rightarrow \infty} \frac{A_{\text{closed}}^{(N; 2m, 2m'; q, q')}}{A_{\text{closed}}^{(N; 0, 0; 0, 0)}} &= \left(-\frac{1}{M_2}\right)^{2(m+m') + q + q'} \left(\frac{1}{2}\right)^{m+m' + q + q'} (2m-1)!!(2m'-1)!! \\ &= \frac{T_{\text{closed}}^{(N; 2m, 2m'; q, q')}}{T_{\text{closed}}^{(N; 0, 0; 0, 0)}}. \end{aligned} \quad (14.62)$$

We now begin to discuss the RR *closed string*, *D-particle* scatterings considered in this chapter.

2. Closed string D-particle scatterings

a. $m = m' = 0$ Case In chapter IX [41], the high energy scattering amplitudes and ratios of fixed angle closed string D-particle scatterings were calculated only for the case $m = m' = 0$. For nonzero m or m' cases, one encounters similar difficulties stated in the paragraph before Eq.(14.61) to calculate the complete fixed angle amplitudes. A subset of ratios can be extracted from Eq.(9.55) and was found to be [41]

$$\frac{T_{SD}^{(N, 0, 0, q, q')}}{T_{SD}^{(N, 0, 0, 0, 0)}} = \left(-\frac{1}{2M_2}\right)^{q+q'}. \quad (14.63)$$

In view of the non-factorizability of Regge string D-particle scattering amplitudes calculated in Eq.(14.49), one is tempted to conjecture that the complete ratios of fixed angle closed string D-particle scatterings may not be factorized. But on the other hand, the decoupling of high energy ZNS seems to imply the factorizability of the fixed angle ratios.

b. General case We can show explicitly that the leading behaviors of the inner products in Eq.(14.41) involving k_1, k_2, e^T, e^P and D are not affected by the replacement of e^P with e^L if we take the limit $b_0 \rightarrow \infty$ after taking the Regge limit. Therefore we proceed as in the previous works on Regge scattering. The calculation for the complete ratios of leading (in b_0) RR *closed string, D-particle* scatterings from Eq.(14.58) gives

$$\begin{aligned}
& \lim_{b_0 \rightarrow \infty} \frac{A_{SD}^{(N;2m,2m';q,q')}}{A_{SD}^{(N,0,0,0)}} \\
&= (i)^{-q-q'} \left(-i \frac{b_0}{2b_0 M_2} \right)^{q+q'} \left(\frac{1}{b_0 M_2^2} \right)^{m+m'} \\
&\cdot \sum_{j=0}^{\min\{2m,2m'\}} (-1)^j j! \binom{2m}{j} \binom{2m'}{j} \left(\frac{b_0}{2} \right)^j \frac{(2m+2m'-2j)!}{(m+m'-j)!} 2^{-2m-2m'+2j} b_0^{m+m'-j} \\
&= (i)^{-q-q'} \left(-i \frac{1}{2M_2} \right)^{q+q'} \left(\frac{1}{2M_2} \right)^{2m+2m'} \\
&\cdot \sum_{j=0}^{\min\{2m,2m'\}} j! \binom{2m}{j} \binom{2m'}{j} (-2)^j \frac{(2m+2m'-2j)!}{(m+m'-j)!}. \tag{14.64}
\end{aligned}$$

In deriving Eq.(14.64), we have made use of Eq.(14.48) and Eq.(14.60). Note that each term in the summation of Eq.(14.64) is not factorized.

Surprisingly, the summation in Eq.(14.64) can be performed, and the ratios can be calculated to be

$$\begin{aligned}
& \lim_{b_0 \rightarrow \infty} \frac{A_{SD}^{(N;2m,2m';q,q')}}{A_{SD}^{(N,0,0,0)}} \\
&= (-)^{q+q'} \left(\frac{1}{2} \right)^{q+q'+2m+2m'} \left(\frac{1}{M_2} \right)^{2m+2m'+q+q'} \\
&\cdot \frac{2^{2m+2m'} \pi \sec \left[\frac{\pi}{2} (2m+2m') \right]}{\Gamma \left(\frac{1-2m}{2} \right) \Gamma \left(\frac{1-2m'}{2} \right)} \\
&= \left(-\frac{1}{M_2} \right)^{2m+q} \left(\frac{1}{2} \right)^{m+q} (2m-1)!! \left(-\frac{1}{M_2} \right)^{2m'+q'} \left(\frac{1}{2} \right)^{m'+q'} (2m'-1)!! \tag{14.65}
\end{aligned}$$

which are *factorized*. They are exactly the same with the ratios of the high energy, fixed angle closed string-string scattering amplitudes calculated in Eq.(14.62) and again consistent

with the decoupling of high energy ZNS in Eq.(5.60) [26–31, 33, 35, 44]. We thus conclude that the identification in Eq.(14.59) continues to work for string D-particle scatterings. So the complete ratios of fixed angle closed string D-particle scatterings are

$$\begin{aligned} \frac{T_{SD}^{(N;2m,2m';q,q')}}{T_{SD}^{(N;0,0;0,0)}} &= \left(-\frac{1}{M_2}\right)^{2(m+m')+q+q'} \left(\frac{1}{2}\right)^{m+m'+q+q'} (2m-1)!!(2m'-1)!! \\ &= \lim_{b_0 \rightarrow \infty} \frac{A_{SD}^{(N;2m,2m';q,q')}}{A_{SD}^{(N,0,0,0)}} \end{aligned} \quad (14.66)$$

where the first equality can be deduced from the decoupling of high energy ZNS. Note that, for $m = m' = 0$, Eq.(14.66) reduces to Eq.(14.63) calculated previously in chapter IX [41].

It is well known that the closed string-string scattering amplitudes can be factorized into two open string-string scattering amplitudes due to the existence of the KLT formula [43]. On the contrary, there is no physical picture for open string D-particle tree scattering amplitudes and thus no factorization for closed string D-particle scatterings into two channels of open string D-particle scatterings, and hence no KLT-like formula there. Here what we really mean is: two string, two D-particle scattering in the limit of infinite D-particle mass. This can also be seen from the nontrivial string D-particle propagator in Eq.(14.6), which vanishes for the case of closed string-string scattering.

Thus the factorized ratios in high energy fixed angle regime calculated in the RR in Eq.(14.65) and Eq.(14.66) came as a surprise. However, these ratios are consistent with the decoupling of high energy ZNS calculated previously in Eq.(5.60)[26–31, 33, 35, 44]. It will be interesting if one can calculate the complete fixed angle amplitudes directly and see how the non-factorized amplitudes can give the result of factorized ratios.

XV. THE APPELL FUNCTIONS F_1 AND THE COMPLETE RSSA

In this chapter, we will show that [73] each 26D open bosonic Regge string scattering amplitude (RSSA) can be expressed in terms of one single Appell function F_1 in the Regge limit. This result enables us to derive infinite number of recurrence relations among RSSA at arbitrary mass levels, which are conjectured to be related to the known $SL(5, C)$ dynamical symmetry of F_1 . Since there is only one single Appell function in the expression of the amplitudes in contrast to a sum of Kummer functions discussed in chapter XI, it is easier to systematically construct recurrence relations among RSSA by directly using recurrence

relations of Appell functions.

In addition, we show that these recurrence relations in the Regge limit can be systematically solved so that all RSSA can be expressed in terms of one amplitude. All these results are dual to high energy symmetries of fixed angle string scattering amplitudes discovered previously in chapter V[26–31, 33, 35, 44].

A. Appell functions and RSSA

The leading order high energy open string states in the Regge regime at each fixed mass level $N = \sum_{n,m,l>0} np_n + mq_m + lr_l$ are [70, 73]

$$|p_n, q_m, r_l\rangle = \prod_{n>0} (\alpha_{-n}^T)^{p_n} \prod_{m>0} (\alpha_{-m}^P)^{q_m} \prod_{l>0} (\alpha_{-l}^L)^{r_l} |0, k\rangle. \quad (15.1)$$

The momenta of the four particles on the scattering plane are

$$k_1 = \left(+\sqrt{p^2 + M_1^2}, -p, 0 \right), \quad (15.2)$$

$$k_2 = \left(+\sqrt{p^2 + M_2^2}, +p, 0 \right), \quad (15.3)$$

$$k_3 = \left(-\sqrt{q^2 + M_3^2}, -q \cos \phi, -q \sin \phi \right), \quad (15.4)$$

$$k_4 = \left(-\sqrt{q^2 + M_4^2}, +q \cos \phi, +q \sin \phi \right) \quad (15.5)$$

where $p \equiv |\tilde{p}|$, $q \equiv |\tilde{q}|$ and $k_i^2 = -M_i^2$. The relevant kinematics in the Regge regime are

$$e^P \cdot k_1 \simeq -\frac{s}{2M_2}, \quad e^P \cdot k_3 \simeq -\frac{\tilde{t}}{2M_2} = -\frac{t - M_2^2 - M_3^2}{2M_2}; \quad (15.6)$$

$$e^L \cdot k_1 \simeq -\frac{s}{2M_2}, \quad e^L \cdot k_3 \simeq -\frac{\tilde{t}'}{2M_2} = -\frac{t + M_2^2 - M_3^2}{2M_2}; \quad (15.7)$$

$$e^T \cdot k_1 = 0, \quad e^T \cdot k_3 \simeq -\sqrt{-t} \quad (15.8)$$

where $\tilde{t} = t - M_2^2 - M_3^2$ and $\tilde{t}' = t + M_2^2 - M_3^2$. The $s - t$ channel one higher spin and three tachyons string scattering amplitudes in the Regge limit can be calculated as

$$\begin{aligned}
A^{(p_n; q_m; r_l)} &= \int_0^1 dx x^{k_1 \cdot k_2} (1-x)^{k_2 \cdot k_3} \cdot \left[\frac{e^P \cdot k_1}{x} - \frac{e^P \cdot k_3}{1-x} \right]^{q_1} \left[\frac{e^L \cdot k_1}{x} + \frac{e^L \cdot k_3}{1-x} \right]^{r_1} \\
&\cdot \prod_{n=1} \left[\frac{(n-1)! e^T \cdot k_3}{(1-x)^n} \right]^{p_n} \prod_{m=2} \left[\frac{(m-1)! e^P \cdot k_3}{(1-x)^m} \right]^{q_m} \prod_{l=2} \left[\frac{(l-1)! e^L \cdot k_3}{(1-x)^l} \right]^{r_l} \\
&= \prod_{n=1} [(n-1)! \sqrt{-t}]^{p_n} \prod_{m=1} \left[-(m-1)! \frac{\tilde{t}}{2M_2} \right]^{q_m} \prod_{l=1} \left[(l-1)! \frac{\tilde{t}'}{2M_2} \right]^{r_l} \\
&\cdot \sum_{j=0}^{r_1} \sum_{i=0}^{q_1} \binom{r_1}{j} \binom{q_1}{i} \left(-\frac{s}{\tilde{t}} \right)^i \left(-\frac{s}{\tilde{t}'} \right)^j B \left(-\frac{s}{2} + N - 1 - i - j, -\frac{t}{2} - 1 + i + j \right)
\end{aligned} \tag{15.9}$$

where in the Regge limit the beta function B can be further reduced to

$$\begin{aligned}
&B \left(-\frac{s}{2} - 1 + N - i - j, -\frac{t}{2} - 1 + i + j \right) \\
&\simeq B \left(-\frac{s}{2} - 1, -\frac{t}{2} - 1 \right) \frac{(-1)^{i+j} \left(-\frac{t}{2} - 1 \right)_{i+j}}{\left(\frac{s}{2} \right)_{i+j}}.
\end{aligned} \tag{15.10}$$

Thus

$$\begin{aligned}
A^{(p_n; q_m; r_l)} &= B \left(-\frac{s}{2} - 1, -\frac{t}{2} - 1 \right) \\
&\cdot \prod_{n=1} [(n-1)! \sqrt{-t}]^{p_n} \prod_{m=1} \left[-(m-1)! \frac{\tilde{t}}{2M_2} \right]^{q_m} \prod_{l=1} \left[(l-1)! \frac{\tilde{t}'}{2M_2} \right]^{r_l} \\
&\cdot \sum_{j=0}^{r_1} \sum_{i=0}^{q_1} \binom{r_1}{j} \binom{q_1}{i} \frac{\left(-\frac{t}{2} - 1 \right)_{i+j}}{\left(\frac{s}{2} \right)_{i+j}} \left(\frac{s}{\tilde{t}} \right)^i \left(\frac{s}{\tilde{t}'} \right)^j
\end{aligned} \tag{15.11}$$

in which the double summation can be expressed in terms of the Appell function F_1 as

$$\begin{aligned}
&\sum_{j=0}^{r_1} \sum_{i=0}^{q_1} \binom{r_1}{j} \binom{q_1}{i} \frac{\left(-\frac{t}{2} - 1 \right)_{i+j}}{\left(\frac{s}{2} \right)_{i+j}} \left(\frac{s}{\tilde{t}} \right)^i \left(\frac{s}{\tilde{t}'} \right)^j \\
&= \sum_{j=0}^{r_1} \sum_{i=0}^{q_1} \frac{(-q_1)_i (-r_1)_j}{i! j!} \frac{\left(-\frac{t}{2} - 1 \right)_{i+j}}{\left(\frac{s}{2} \right)_{i+j}} \left(-\frac{s}{\tilde{t}} \right)^i \left(-\frac{s}{\tilde{t}'} \right)^j \\
&= F_1 \left(-\frac{t}{2} - 1; -q_1, -r_1; \frac{s}{2}; -\frac{s}{\tilde{t}}, -\frac{s}{\tilde{t}'} \right).
\end{aligned} \tag{15.12}$$

The Appell function F_1 is one of the four extensions of the hypergeometric function ${}_2F_1$ to two variables and is defined to be

$$F_1(a; b, b'; c; x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a)_{m+n} (b)_m (b')_n}{m! n! (c)_{m+n}} x^m y^n \tag{15.13}$$

where $(a)_n = a \cdot (a+1) \cdots (a+n-1)$ is the rising Pochhammer symbol. Note that when a or $b(b')$ is a non-positive integer, the Appell function truncates to a polynomial. This is the case for the Appell function in the RSSA calculated in Eq.(15.14) in the following

$$A^{(p_n; q_m; r_l)} = \prod_{n=1} [(n-1)! \sqrt{-t}]^{p_n} \prod_{m=1} \left[-(m-1)! \frac{\tilde{t}}{2M_2} \right]^{q_m} \prod_{l=1} \left[(l-1)! \frac{\tilde{t}'}{2M_2} \right]^{r_l} \cdot F_1 \left(-\frac{t}{2} - 1; -q_1, -r_1; \frac{s}{2}; -\frac{s}{\tilde{t}}, -\frac{s}{\tilde{t}'} \right) \cdot B \left(-\frac{s}{2} - 1, -\frac{t}{2} - 1 \right). \quad (15.14)$$

Alternatively, it is interesting to note that the result calculated in Eq.(15.14) can be directly obtained from an integral representation of F_1 due to Emile Picard (1881) [148]

$$F_1(a; b_1, b_2; c; x, y) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 dt t^{a-1} (1-t)^{c-a-1} (1-xt)^{-b_1} (1-yt)^{-b_2}, \quad (15.15)$$

which was later generalized by Appell and Kampe de Fariet (1926) [149] to n variables

$$F_1(a; b_1, b_2, \dots, b_n; c; x_1, x_2, \dots, x_n) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 dt t^{a-1} (1-t)^{c-a-1} \cdot (1-x_1 t)^{-b_1} (1-x_2 t)^{-b_2} \dots (1-x_n t)^{-b_n}. \quad (15.16)$$

Eq.(15.16) may have application for higher point RSSA. To apply the Picard formula in Eq.(15.15), we do the transformation $x \rightarrow (1-x)$, and RSSA can be calculated to be

$$\begin{aligned} A^{(p_n; q_m; r_l)} &= \int_0^1 dx (1-x)^{-\frac{s}{2}+N-2} x^{-\frac{t}{2}-2} \cdot \left[1 - \frac{s}{\tilde{t}} \frac{x}{1-x} \right]^{q_1} \left[1 - \frac{s}{\tilde{t}'} \frac{x}{1-x} \right]^{r_1} \\ &\cdot \prod_{n=1} [(n-1)! \sqrt{-t}]^{p_n} \prod_{m=1} \left[-(m-1)! \frac{\tilde{t}}{2M_2} \right]^{q_m} \prod_{l=1} \left[(l-1)! \frac{\tilde{t}'}{2M_2} \right]^{r_l} \\ &\simeq B \left(-\frac{t}{2} - 1, -\frac{s}{2} - 1 \right) \cdot F_1 \left(-\frac{t}{2} - 1, -q_1, -r_1, -\frac{s}{2}; \frac{s}{\tilde{t}}, \frac{s}{\tilde{t}'} \right) \\ &\cdot \prod_{n=1} [(n-1)! \sqrt{-t}]^{p_n} \prod_{m=1} \left[-(m-1)! \frac{\tilde{t}}{2M_2} \right]^{q_m} \prod_{l=1} \left[(l-1)! \frac{\tilde{t}'}{2M_2} \right]^{r_l}, \quad (15.17) \end{aligned}$$

which is consistent with the result calculated in Eq.(15.14). It is important to note that although F_1 in Eq.(15.14) is a polynomial in s , the result in Eq.(15.14) is valid only for the *leading order* in s in the Regge limit. Note that in contrast to the previous calculation [70] in Eq.(11.86) and Eq.(11.87) where a finite sum of Kummer functions was obtained, here we get only one single Appell function in Eq.(15.14). This simplification will greatly simplify the calculation of recurrence relations among RSSA to be discussed in the next section.

B. Solving all RSSA by Appell recurrence relations

The Appell function F_1 entails four recurrence relations among contiguous functions

$$(a - b_1 - b_2) F_1(a; b_1, b_2; c; x, y) - a F_1(a + 1; b_1, b_2; c; x, y) + b_1 F_1(a; b_1 + 1, b_2; c; x, y) + b_2 F_1(a; b_1, b_2 + 1; c; x, y) = 0, \quad (15.18)$$

$$c F_1(a; b_1, b_2; c; x, y) - (c - a) F_1(a; b_1, b_2; c + 1; x, y) - a F_1(a + 1; b_1, b_2; c + 1; x, y) = 0, \quad (15.19)$$

$$c F_1(a; b_1, b_2; c; x, y) + c(x - 1) F_1(a; b_1 + 1, b_2; c; x, y) - (c - a)x F_1(a; b_1 + 1, b_2; c + 1; x, y) = 0, \quad (15.20)$$

$$c F_1(a; b_1, b_2; c; x, y) + c(y - 1) F_1(a; b_1, b_2 + 1; c; x, y) - (c - a)y F_1(a; b_1, b_2 + 1; c + 1; x, y) = 0. \quad (15.21)$$

All other recurrence relations can be deduced from these four relations. We can easily solve the Appell function in Eq.(15.14) and express it in terms of the RSSA

$$F_1\left(-\frac{t}{2} - 1; -q_1, -r_1; \frac{s}{2}; -\frac{s}{\tilde{t}}, -\frac{s}{\tilde{t}'}\right) = \frac{A^{(p_n; q_m; r_l)}}{B\left(-\frac{s}{2} - 1, -\frac{t}{2} - 1\right)} \prod_{n=1} [(n-1)! \sqrt{-t}]^{-p_n} \prod_{m=1} \left[-(m-1)! \frac{\tilde{t}}{2M_2} \right]^{-q_m} \prod_{l=1} \left[(l-1)! \frac{\tilde{t}'}{2M_2} \right]^{-r_l}. \quad (15.22)$$

Note that among the set of integers (p_n, q_m, r_l) on the right hand side of Eq.(15.22), only $(-q_1, -r_1)$ dependence shows up on the Appell function F_1 on the left hand side of Eq.(15.22).

Indeed, for those highest spin string states at the mass level $M_2^2 = 2(N - 1)$

$$|N; q_1, r_1\rangle \equiv (\alpha_{-1}^T)^{N-q_1-r_1} (\alpha_{-1}^P)^{q_1} (\alpha_{-1}^L)^{r_1} |0, k\rangle, \quad (15.23)$$

the string amplitudes reduce to

$$A^{(N; q_1, r_1)} = (\sqrt{-t})^{N-q_1-r_1} \left(-\frac{\tilde{t}}{2M_2}\right)^{q_1} \left(\frac{\tilde{t}'}{2M_2}\right)^{r_1} \cdot F_1\left(-\frac{t}{2} - 1; -q_1, -r_1; \frac{s}{2}; -\frac{s}{\tilde{t}}, -\frac{s}{\tilde{t}'}\right) B\left(-\frac{s}{2} - 1, -\frac{t}{2} - 1\right), \quad (15.24)$$

which can be used to solve easily the Appell function F_1 in terms of the RSSA $A^{(N; q_1, r_1)}$.

We now proceed to show that the recurrence relations of the Appell function F_1 in the *Regge limit* in Eq.(15.14) can be systematically solved so that all RSSA can be expressed in

terms of one amplitude. As the first step, we note that in [70] the RSSA was expressed in terms of finite sum of Kummer functions. There are two equivalent expressions [70] which were written in Eq.(11.86) and Eq.(11.87) in section XI.D.

It is easy to see that, for $q_1 = 0$ in Eq.(11.86) or $r_1 = 0$ in Eq.(11.87), the RSSA can be expressed in terms of only one single Kummer function $U\left(-r_1, \frac{t}{2} + 2 - r_1, \frac{\tilde{t}}{2}\right)$ or $U\left(-q_1, \frac{t}{2} + 2 - q_1, \frac{\tilde{t}}{2}\right)$, which are thus related to the Appell function $F_1\left(-\frac{t}{2} - 1; 0, -r_1; \frac{s}{2}; -\frac{s}{t}, -\frac{s}{\tilde{t}}\right)$ or $F_1\left(-\frac{t}{2} - 1; -q_1, 0; \frac{s}{2}; -\frac{s}{t}, -\frac{s}{\tilde{t}}\right)$ respectively in the Regge limit in Eq.(15.14). Indeed, one can easily calculate

$$\lim_{s \rightarrow \infty} F_1\left(-\frac{t}{2} - 1; 0, -r_1; \frac{s}{2}; -\frac{s}{t}, -\frac{s}{\tilde{t}}\right) = \left(\frac{2}{\tilde{t}}\right)^{r_1} U\left(-r_1, \frac{t}{2} + 2 - r_1, \frac{\tilde{t}}{2}\right), \quad (15.25)$$

$$\lim_{s \rightarrow \infty} F_1\left(-\frac{t}{2} - 1; -q_1, 0; \frac{s}{2}; -\frac{s}{t}, -\frac{s}{\tilde{t}}\right) = \left(\frac{2}{\tilde{t}}\right)^{q_1} U\left(-q_1, \frac{t}{2} + 2 - q_1, \frac{\tilde{t}}{2}\right). \quad (15.26)$$

On the other hand, it was shown in Eq.(11.141) [70] that the ratio

$$\frac{U(\alpha, \gamma, z)}{U(0, z, z)} = f(\alpha, \gamma, z), \alpha = 0, -1, -2, -3, \dots \quad (15.27)$$

is determined and $f(\alpha, \gamma, z)$ can be calculated by using recurrence relations of $U(\alpha, \gamma, z)$. Note that $U(0, z, z) = 1$ by explicit calculation. We thus conclude that in the Regge limit

$$c = \frac{s}{2} \rightarrow \infty; x, y \rightarrow \infty; a, b_1, b_2 \text{ fixed}, \quad (15.28)$$

the Appell functions $F_1(a; 0, b_2; c; x, y)$ and $F_1(a; b_1, 0; c; x, y)$ are determined up to an overall factor by recurrence relations. The next step is to derive the recurrence relation

$$yF_1(a; b_1, b_2; c; x, y) - xF_1(a; b_1 + 1, b_2 - 1; c; x, y) + (x - y)F_1(a; b_1 + 1, b_2; c; x, y) = 0, \quad (15.29)$$

which can be obtained from Eq.(15.20) and Eq.(15.21). We are now ready to show that the recurrence relations of the Appell function F_1 in the Regge limit in Eq.(15.14) can be systematically solved so that all RSSA can be expressed in terms of one amplitude. We will use the short notation $F_1(a; b_1, b_2; c; x, y) = F_1(b_1, b_2)$ in the following. For $b_2 = -1$, by using Eq.(15.29) and the known $F_1(b_1, 0)$ and $F_1(0, b_2)$, one can easily show that $F_1(b_1, -1)$ are determined for all $b_1 = -1, -2, -3, \dots$. Similarly, $F_1(b_1, -2)$ are determined for all $b_1 = -1, -2, -3, \dots$ if one uses the result of $F_1(b_1, -1)$ in addition to Eq.(15.29) and the known $F_1(b_1, 0)$ and $F_1(0, b_2)$. This process can be continued and one ends up with the result that $F_1(b_1, b_2)$ are determined for all $b_1, b_2 = -1, -2, -3, \dots$. This completes the proof that

the recurrence relations of the Appell function F_1 in the Regge limit in Eq.(15.14) can be systematically solved so that all RSSA can be expressed in terms of one amplitude.

C. Higher recurrence relations

With the result calculated in Eq.(15.14), one can easily derive many recurrence relations among RSSA at arbitrary mass levels. For example, the identity in Eq.(15.29) leads to

$$\sqrt{-t} [A^{(N;q_1,r_1)} + A^{(N;q_1-1,r_1+1)}] - M_2 A^{(N;q_1-1,r_1)} = 0, \quad (15.30)$$

which is the generalization of Eq.(11.152) discussed in chapter XI [70] for mass level $M_2^2 = 4$ to arbitrary mass levels $M_2^2 = 2(N-1)$. Incidentally, one should keep in mind that the recurrence relations among RSSA are valid only in the Regge limit. We give one example to illustrate the calculation. By using Eq.(15.18) and Eq.(15.19), we have

$$\begin{aligned} & (c - b_1 - b_2) F_1(a; b_1, b_2; c+1; x, y) - c F_1(a; b_1, b_2; c; x, y) \\ & + b_1 F_1(a; b_1+1, b_2; c+1; x, y) + b_2 F_1(a; b_1, b_2+1; c+1; x, y) = 0. \end{aligned} \quad (15.31)$$

Then with Eq.(15.20) and Eq.(15.21), we obtain

$$\begin{aligned} & (c - b_1 - b_2) y F_1(a; b_1-1, b_2; c; x, y) \\ & + [(a - b_1 - b_2) xy - (c - 2b_1 - b_2) y + b_2 x] F_1(a; b_1, b_2; c; x, y) \\ & + b_1 (x-1) y F_1(a; b_1+1, b_2; c; x, y) + b_2 x (y-1) F_1(a; b_1, b_2+1; c; x, y) = 0, \end{aligned} \quad (15.32)$$

$$\begin{aligned} & (c - b_1 - b_2) x F_1(a; b_1, b_2-1; c; x, y) \\ & + [(a - b_1 - b_2) xy - (c - b_1 - 2b_2) x + b_1 y] F_1(a; b_1, b_2; c; x, y) \\ & + b_1 (x-1) y F_1(a; b_1+1, b_2; c; x, y) + b_2 x (y-1) F_1(a; b_1, b_2+1; c; x, y) = 0. \end{aligned} \quad (15.33)$$

Finally by Combining Eq.(15.29) and Eq.(15.33), and taking the leading term of s in the Regge limit, we end up with the recurrence relation for b_2

$$\begin{aligned} & cx^2 F_1(a; b_1, b_2; c; x, y) \\ & + [(a - b_1 - b_2 - 1) xy^2 + cx^2 - 2cxy] F_1(a; b_1, b_2+1; c; x, y) \\ & - [(a+1) x^2 y - (a - b_2 - 1) xy^2 - cx^2 + cxy] F_1(a; b_1, b_2+2; c; x, y) \\ & - (b_2+2) x (x-y) y F_1(a; b_1, b_2+3; c; x, y) = 0, \end{aligned} \quad (15.34)$$

which leads to a recurrence relation for RSSA at arbitrary mass levels

$$\begin{aligned}
& \tilde{t}'^2 A^{(N;q_1,r_1)} \\
& + [\tilde{t}'^2 + \tilde{t} (t - 2\tilde{t}' - 2q_1 - 2r_1 + 4)] \left(\frac{\tilde{t}'}{\sqrt{-t}} \right) A^{(N;q_1,r_1+1)} \\
& + [\tilde{t}'^2 - \tilde{t}' (\tilde{t} + t) + \tilde{t} (t - 2r_1 + 4)] \left(\frac{\tilde{t}'}{\sqrt{-t}} \right)^2 A^{(N;q_1,r_1+2)} \\
& - 2(r_1 - 2) (\tilde{t}' - \tilde{t}) \left(\frac{\tilde{t}'}{\sqrt{-t}} \right)^3 A^{(N;q_1,r_1+3)} = 0. \tag{15.35}
\end{aligned}$$

More higher recurrence relations which contain general number of $l \geq 3$ Appell functions can be found in [75].

Since it was shown that [74] the Appell function F_1 are basis vectors for models of irreducible representations of $sl(5, C)$ algebra, it is reasonable to believe that the spacetime symmetry of Regge string theory is closely related to $SL(5, C)$ non-compact group. In particular, the recurrence relations of RSSA studied in this chapter are related to the $SL(5, C)$ group as well. Further investigation remains to be done and more evidences need to be uncovered.

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Appendix A: Linear relations at mass levels $M^2 = 6$ and $M^2 = 8$ in GR

At mass level $M^2 = 6$, the most general form of physical states at mass level $M^2 = 6$ are given by

$$\begin{aligned}
& [\epsilon_{\mu\nu\lambda\sigma} \alpha_{-1}^\mu \alpha_{-1}^\nu \alpha_{-1}^\lambda \alpha_{-1}^\sigma + \epsilon_{(\mu\nu\lambda)} \alpha_{-1}^\mu \alpha_{-1}^\nu \alpha_{-2}^\lambda + \epsilon_{\mu\nu,\lambda} \alpha_{-1}^\mu \alpha_{-1}^\nu \alpha_{-2}^\lambda \\
& + \epsilon_{(\mu\nu)}^{(1)} \alpha_{-1}^\mu \alpha_{-3}^\nu + \epsilon_{[\mu\nu]}^{(1)} \alpha_{-1}^\mu \alpha_{-3}^\nu + \epsilon_{(\mu\nu)}^{(2)} \alpha_{-2}^\mu \alpha_{-2}^\nu + \epsilon_\mu \alpha_{-4}^\mu] |0, k\rangle, \tag{A.1}
\end{aligned}$$

where $\epsilon_{\mu\nu,\lambda}$ represents the mixed symmetric spin three states, that is, one first symmetrizes $\mu\nu$ and then anti-symmetrizes $\mu\lambda$. The Virasoro constraints are calculated to be

$$2k^\sigma \epsilon_{(\mu\nu\lambda\sigma)} + \epsilon_{(\mu\nu\lambda)} = 0, \quad (\text{A.2})$$

$$2k^\lambda \epsilon_{(\mu\nu\lambda)} + k^\lambda (\epsilon_{\lambda\mu,\nu} + \epsilon_{\mu\lambda,\nu}) + 3(\epsilon_{(\mu\nu)}^{(1)} + \epsilon_{[\mu\nu]}^{(1)}) + 4\epsilon_{(\mu\nu)}^{(2)} = 0, \quad (\text{A.3})$$

$$k^\mu \epsilon_{(\mu\nu)}^{(1)} + k^\mu \epsilon_{[\mu\nu]}^{(1)} + 4\epsilon_\nu = 0, \quad (\text{A.4})$$

$$6\eta^{\lambda\sigma} \epsilon_{(\mu\nu\lambda\sigma)} + 2k^\lambda \epsilon_{(\mu\nu\lambda)} + \frac{1}{2}k^\lambda (\epsilon_{\mu\nu,\lambda} + \epsilon_{\nu\mu,\lambda}) + 3\epsilon_{(\mu\nu)}^{(1)} = 0, \quad (\text{A.5})$$

$$\eta^{\mu\nu} \epsilon_{(\mu\nu\lambda)} + \eta^{\mu\nu} \epsilon_{(\mu\nu,\lambda)} + 4k^\mu \epsilon_{(\mu\nu)}^{(2)} + 4\epsilon_\lambda = 0. \quad (\text{A.6})$$

In the high energy limit, by replacing P by L and ignoring irrelevant states, one gets

$$\epsilon_{(TTTT)} : \epsilon_{(TTLL)} : \epsilon_{(LLLL)} : \epsilon_{TT,L} : \epsilon_{(TTL)} : \epsilon_{(LLL)} : \epsilon_{(LL)}^{(2)} = 48 : 4 : 1 : 12\sqrt{6} : 8\sqrt{6} : 2\sqrt{6} : 6.$$

After including the normalization factor of the field variables and the appropriate symmetry factors, one ends up with

$$\begin{aligned} & \mathcal{T}_{(TTTT)} : \mathcal{T}_{(TTLL)} : \mathcal{T}_{(LLLL)} : \mathcal{T}_{TT,L} : \mathcal{T}_{(TTL)} : \mathcal{T}_{(LLL)} : \mathcal{T}_{(LL)} \\ &= 4!\epsilon_{(TTTT)} : 4!\epsilon_{(TTLL)} : 4!\epsilon_{(LLLL)} : -4\epsilon_{TT,L} : -4\epsilon_{(TTL)} : -4\epsilon_{(LLL)} : 8\epsilon_{(LL)}^{(2)} \\ &= 16 : \frac{4}{3} : \frac{1}{3} : -\frac{2\sqrt{6}}{3} : -\frac{4\sqrt{6}}{9} : -\frac{\sqrt{6}}{9} : \frac{2}{3}. \end{aligned} \quad (\text{A.7})$$

At mass level $M^2 = 8$, the most general form of physical states at mass level $M^2 = 8$ are given by (for simplicity, we neglect terms containing α_{-n}^μ with $n \geq 3$)

$$\begin{aligned} & [\epsilon_{\mu\nu\lambda\sigma\rho} \alpha_{-1}^\mu \alpha_{-1}^\nu \alpha_{-1}^\lambda \alpha_{-1}^\sigma \alpha_{-1}^\rho + \epsilon_{(\mu\nu\lambda\sigma)} \alpha_{-1}^\mu \alpha_{-1}^\nu \alpha_{-1}^\lambda \alpha_{-2}^\rho + \epsilon_{(\mu\nu\lambda)} \alpha_{-1}^\mu \alpha_{-2}^\nu \alpha_{-2}^\lambda \\ & + \epsilon_{\mu\nu\lambda,\sigma} \alpha_{-1}^\mu \alpha_{-1}^\nu \alpha_{-1}^\lambda \alpha_{-2}^\rho + \epsilon_{\mu,\nu\lambda} \alpha_{-1}^\mu \alpha_{-2}^\nu \alpha_{-2}^\lambda] |0, k\rangle, \end{aligned} \quad (\text{A.8})$$

where $\epsilon_{\mu\nu\lambda,\sigma}$ represents the mixed symmetric spin four states, that is, first symmetrizes $\mu\nu\lambda$ and then anti-symmetrizes $\mu\sigma$. Similar definition for the mixed symmetric spin three states

$\epsilon_{\mu,\nu\lambda}$. The Virasoro constraints are calculated to be

$$5k^\sigma \epsilon_{(\mu\nu\lambda\sigma\rho)} + 2\epsilon_{(\mu\nu\lambda\sigma)} = 0, \quad (\text{A.9})$$

$$3k^\lambda \epsilon_{(\mu\nu\lambda\sigma)} + \frac{1}{2}k^\lambda [(\epsilon_{\mu\nu\lambda,\sigma} + \epsilon_{\lambda\mu\nu,\sigma} + \epsilon_{\mu\lambda\nu,\sigma}) + (\mu \leftrightarrow \nu)] \\ + 4\epsilon_{(\mu\nu\sigma)} + \epsilon_{\mu,\nu\sigma} + \epsilon_{\nu,\mu\sigma} = 0, \quad (\text{A.10})$$

$$k^\mu \epsilon_{(\mu\nu\lambda)} + \frac{1}{2}k^\mu (\epsilon_{\mu,\nu\lambda} + \epsilon_{\mu,\lambda\nu}) = 0, \quad (\text{A.11})$$

$$5\eta^{\rho\sigma} \epsilon_{(\mu\nu\lambda\sigma\rho)} + k^\sigma \epsilon_{(\mu\nu\lambda\sigma)} + \frac{1}{3}k^\sigma (\epsilon_{\mu\nu\lambda,\sigma} + \epsilon_{\nu\lambda\mu,\sigma} + \epsilon_{\lambda\mu\nu,\sigma}) = 0, \quad (\text{A.12})$$

$$3\eta^{\nu\lambda} \epsilon_{(\mu\nu\lambda\sigma)} + \eta^{\nu\lambda} (\epsilon_{\mu\nu\lambda,\sigma} + \epsilon_{\lambda\mu\nu,\sigma} + \epsilon_{\nu\lambda\mu,\sigma}) + 4k^\lambda \epsilon_{(\mu\sigma\lambda)} + 2k^\lambda (\epsilon_{\mu,\sigma\lambda} + \epsilon_{\mu,\lambda\sigma}) = 0. \quad (\text{A.13})$$

In the high energy limit, by replacing P by L and ignoring irrelevant states, one gets

$$\epsilon_{(TTTTT)} : \epsilon_{(TTTTL)} : \epsilon_{(TTTLL)} : \epsilon_{(TLLL)} : \epsilon_{(TLLLL)} : \epsilon_{(TLL)} : \epsilon_{T,LL} : \epsilon_{TLL,L} : \epsilon_{TTT,L} \\ = \frac{4}{15} : \frac{\sqrt{2}}{12} : \frac{2}{120} : \frac{\sqrt{2}}{64} : \frac{1}{320} : \frac{1}{24} : \frac{1}{12} : \frac{\sqrt{2}}{192} : \frac{\sqrt{2}}{4}. \quad (\text{A.14})$$

After including the normalization factor of the field variables and the appropriate symmetry factors, one ends up with

$$\mathcal{T}_{(TTTTT)} : \mathcal{T}_{(TTTTL)} : \mathcal{T}_{(TTTLL)} : \mathcal{T}_{(TLLL)} : \mathcal{T}_{(TLLLL)} : \mathcal{T}_{(TLL)} : \mathcal{T}_{T,LL} : \mathcal{T}_{TLL,L} : \mathcal{T}_{TTT,L} \\ = 5!\epsilon_{(TTTTT)} : 3! \times 2\epsilon_{(TTTTL)} : 5!\epsilon_{(TTTLL)} : 3! \times 2\epsilon_{(TLLL)} : 5!\epsilon_{(TLLLL)} \\ : 8\epsilon_{(TLL)} : 8\epsilon_{T,LL} : 3! \times 2\epsilon_{TLL,L} : 3! \times 2\epsilon_{TTT,L} \\ = 32 : \sqrt{2} : 2 : \frac{3\sqrt{2}}{16} : \frac{3}{8} : \frac{1}{3} : \frac{2}{3} : \frac{\sqrt{2}}{16} : 3\sqrt{2}. \quad (\text{A.15})$$

Appendix B: High energy limit of Virasoro constraints

1. Bosonic String

To take the high energy limit for the Virasoro constraints, we replace the indices (μ_i, ν_i) by L or T , and

$$k^{\mu_i} \rightarrow M e^L, \eta^{\mu_1\mu_2} \rightarrow e^T e^T. \quad (\text{B.1})$$

where M is the mass operator. Equations (5.110a) and (5.110b) become

$$\begin{aligned}
0 = & M \left[L \left| \mu_2^1 \cdots \mu_{m_1}^1 \right. \right] \bigotimes_{j \neq 1}^N \left[\mu_1^j \cdots \mu_{m_j}^j \right] \\
& + \sum_{i=2}^{m_1} \left[\mu_2^1 \cdots \hat{\mu}_i^1 \cdots \mu_{m_1}^1 \right] \otimes \left[\mu_i^1 \mu_1^2 \cdots \mu_{m_2}^2 \right] \bigotimes_{j \neq 1,2}^N \left[\mu_1^j \cdots \mu_{m_j}^j \right] \\
& + \sum_{l=3}^N (l-1) \left[\mu_2^1 \cdots \mu_{m_1}^1 \right] \\
& \otimes \sum_{i=1}^{m_{l-1}} \left[\mu_1^{l-1} \cdots \hat{\mu}_i^{l-1} \cdots \mu_{m_{l-1}}^{l-1} \right] \otimes \left[\mu_i^{l-1} \mu_1^l \cdots \mu_{m_l}^l \right] \bigotimes_{j \neq 1, l, l-1}^N \left[\mu_1^j \cdots \mu_{m_j}^j \right], \tag{B.2a}
\end{aligned}$$

and

$$\begin{aligned}
0 = & \frac{1}{2} \left[T \left| T \mu_3^1 \cdots \mu_{m_1}^1 \right. \right] \bigotimes_{j \neq 1}^N \left[\mu_1^j \cdots \mu_{m_j}^j \right] \\
& + M \left[\mu_3^1 \cdots \mu_{m_1}^1 \right] \otimes \left[\mu_1^2 \cdots \mu_{m_2}^2 \right] L \bigotimes_{j \neq 1,2}^N \left[\mu_1^j \cdots \mu_{m_j}^j \right] \\
& + \sum_{i=3}^{m_1} \left[\mu_3^1 \cdots \hat{\mu}_i^1 \cdots \mu_{m_1}^1 \right] \otimes \left[\mu_i^1 \mu_1^3 \cdots \mu_{m_3}^3 \right] \bigotimes_{j \neq 1,3}^N \left[\mu_1^j \cdots \mu_{m_j}^j \right] \\
& + \sum_{l=4}^N (l-2) \left[\mu_3^1 \cdots \mu_{m_1}^1 \right] \\
& \otimes \sum_{i=1}^{m_{l-2}} \left[\mu_1^{l-2} \cdots \hat{\mu}_i^{l-2} \cdots \mu_{m_l}^{l-2} \right] \otimes \left[\mu_i^{l-2} \mu_1^l \cdots \mu_{m_l}^l \right] \bigotimes_{j \neq 1, l, l-2}^N \left[\mu_1^j \cdots \mu_{m_j}^j \right]. \tag{B.2b}
\end{aligned}$$

The indices and $\{\mu_i^j\}$ are symmetric and can be chosen to have l_j of $\{L\}$ which $0 \leq l_j \leq m_j$ and $\{T\}$ for the rest. Thus

$$\begin{aligned}
0 = & M \underbrace{\mu_2^1 T \cdots T L \cdots L}_{m_1-2-l_1 \quad l_1+1} \overset{N}{\otimes}_{j \neq 1} \underbrace{\mu_1^j T \cdots T L \cdots L}_{m_j-1-l_j \quad l_j} \\
& + \underbrace{T \cdots T L \cdots L}_{m_1-2-l_1 \quad l_1} \otimes \underbrace{\mu_2^1 \mu_1^2 T \cdots T L \cdots L}_{m_2-1-l_2 \quad l_2} \overset{N}{\otimes}_{j \neq 1,2} \underbrace{\mu_1^j T \cdots T L \cdots L}_{m_j-1-l_j \quad l_j} \\
& + (m_1 - 2 - l_1) \underbrace{\mu_2^1 T \cdots T L \cdots L}_{m_1-3-l_1 \quad l_1} \otimes \underbrace{\mu_1^2 T \cdots T L \cdots L}_{m_2-l_2 \quad l_2} \overset{N}{\otimes}_{j \neq 1,2} \underbrace{\mu_1^j T \cdots T L \cdots L}_{m_j-1-l_j \quad l_j} \\
& + l_1 \underbrace{\mu_2^1 T \cdots T L \cdots L}_{m_1-2-l_1 \quad l_1-1} \otimes \underbrace{\mu_1^2 T \cdots T L \cdots L}_{m_2-1-l_2 \quad l_2+1} \overset{N}{\otimes}_{j \neq 1,2} \underbrace{\mu_1^j T \cdots T L \cdots L}_{m_j-1-l_j \quad l_j} \\
& + \sum_{l=3}^N (l-1) \underbrace{\mu_2^1 T \cdots T L \cdots L}_{m_1-2-l_1 \quad l_1} \otimes \underbrace{T \cdots T L \cdots L}_{m_{l-1}-1-l_{l-1} \quad l_{l-1}} \\
& \otimes \underbrace{\mu_1^{l-1} \mu_1^l T \cdots T L \cdots L}_{m_l-1-l_l \quad l_l} \overset{N}{\otimes}_{j \neq 1, l, l-1} \underbrace{\mu_1^j T \cdots T L \cdots L}_{m_j-1-l_j \quad l_j} \\
& + \sum_{l=3}^N (l-1) (m_{l-1} - 1 - l_{l-1}) \underbrace{\mu_2^1 T \cdots T L \cdots L}_{m_1-2-l_1 \quad l_1} \otimes \underbrace{\mu_1^{l-1} T \cdots T L \cdots L}_{m_{l-1}-2-l_{l-1} \quad l_{l-1}} \\
& \otimes \underbrace{\mu_1^l T \cdots T L \cdots L}_{m_l-l_l \quad l_l} \overset{N}{\otimes}_{j \neq 1, l, l-1} \underbrace{\mu_1^j T \cdots T L \cdots L}_{m_j-1-l_j \quad l_j} \\
& + \sum_{l=3}^N l_{l-1} (l-1) \underbrace{\mu_2^1 T \cdots T L \cdots L}_{m_1-2-l_1 \quad l_1} \otimes \underbrace{\mu_1^{l-1} T \cdots T L \cdots L}_{m_{l-1}-1-l_{l-1} \quad l_{l-1}-1} \\
& \otimes \underbrace{\mu_1^l T \cdots T L \cdots L}_{m_l-1-l_l \quad l_l+1} \overset{N}{\otimes}_{j \neq 1, l, l-1} \underbrace{\mu_1^j T \cdots T L \cdots L}_{m_j-1-l_j \quad l_j}, \tag{B.3a}
\end{aligned}$$

and

$$\begin{aligned}
0 = & \frac{1}{2} \underbrace{\mu_3^1 T \cdots T L \cdots L}_{m_1-3-l_1 \quad l_1} \otimes_{j \neq 1}^N \underbrace{\mu_1^j T \cdots T L \cdots L}_{m_j-1-l_j \quad l_j} \\
& + M \underbrace{\mu_3^1 T \cdots T L \cdots L}_{m_1-3-l_1 \quad l_1} \otimes \underbrace{\mu_1^2 T \cdots T L \cdots L}_{m_2-1-l_2 \quad l_2+1} \otimes_{j \neq 1,2}^N \underbrace{\mu_1^j T \cdots T L \cdots L}_{m_j-1-l_j \quad l_j} \\
& + \underbrace{T \cdots T L \cdots L}_{m_1-3-l_1 \quad l_1} \otimes \underbrace{\mu_3^1 \mu_1^3 T \cdots T L \cdots L}_{m_3-1-l_3 \quad l_3} \otimes_{j \neq 1,3}^N \underbrace{\mu_1^j T \cdots T L \cdots L}_{m_j-1-l_j \quad l_j} \\
& + (m_1 - 3 - l_1) \underbrace{\mu_3^1 T \cdots T L \cdots L}_{m_1-4-l_1 \quad l_1} \otimes \underbrace{\mu_1^3 T \cdots T L \cdots L}_{m_3-l_3 \quad l_3} \otimes_{j \neq 1,3}^N \underbrace{\mu_1^j T \cdots T L \cdots L}_{m_j-1-l_j \quad l_j} \\
& + l_1 \underbrace{\mu_3^1 T \cdots T L \cdots L}_{m_1-3-l_1 \quad l_1-1} \otimes \underbrace{\mu_1^3 T \cdots T L \cdots L}_{m_3-1-l_3 \quad l_3+1} \otimes_{j \neq 1,3}^N \underbrace{\mu_1^j T \cdots T L \cdots L}_{m_j-1-l_j \quad l_j} \\
& + \sum_{l=4}^N (l-2) \underbrace{\mu_3^1 T \cdots T L \cdots L}_{m_1-3-l_1 \quad l_1} \otimes \underbrace{T \cdots T L \cdots L}_{m_{l-2}-1-l_{l-2} \quad l_{l-2}} \\
& \otimes \underbrace{\mu_1^{l-2} \mu_1^l T \cdots T L \cdots L}_{m_l-1-l_l \quad l_l} \otimes_{j \neq 1,l,l-2}^N \underbrace{\mu_1^j T \cdots T L \cdots L}_{m_j-1-l_j \quad l_j} \\
& + \sum_{l=4}^N (l-2) (m_{l-2} - 1 - l_{l-2}) \underbrace{\mu_3^1 T \cdots T L \cdots L}_{m_1-3-l_1 \quad l_1} \otimes \underbrace{\mu_1^{l-2} T \cdots T L \cdots L}_{m_{l-2}-2-l_{l-2} \quad l_{l-2}} \\
& \otimes \underbrace{\mu_1^l T \cdots T L \cdots L}_{m_l-l_l \quad l_l} \otimes_{j \neq 1,l,l-2}^N \underbrace{\mu_1^j T \cdots T L \cdots L}_{m_j-1-l_j \quad l_j} \\
& + \sum_{l=4}^N l_{l-2} (l-2) \underbrace{\mu_3^1 T \cdots T L \cdots L}_{m_1-3-l_1 \quad l_1} \otimes \sum_{i=2}^{m_{l-2}} \underbrace{\mu_1^{l-2} T \cdots T L \cdots L}_{m_{l-2}-1-l_{l-2} \quad l_{l-2}-1} \\
& \otimes \underbrace{\mu_1^l T \cdots T L \cdots L}_{m_l-1-l_l \quad l_l+1} \otimes_{j \neq 1,l,l-2}^N \underbrace{\mu_1^j T \cdots T L \cdots L}_{m_j-1-l_j \quad l_j}. \tag{B.3b}
\end{aligned}$$

There are still some undetermined parameters μ_2^1 , μ_3^1 and μ_1^j ($j \geq 2$), which can be chosen to be L or T , in the above equations. However, it is easy to see that both choices lead to the same equations. Therefore, we will set all of them to be T in the following. The final

Virasoro constraints at high energies become

$$\begin{aligned}
0 = & M \underbrace{\boxed{T \cdots T L \cdots L}}_{m_1-1-l_1} \underbrace{\quad}_{l_1+1} \bigotimes_{j \neq 1}^N \underbrace{\boxed{T \cdots T L \cdots L}}_{m_j-l_j} \underbrace{\quad}_{l_j} \\
& + (m_1 - 1 - l_1) \underbrace{\boxed{T \cdots T L \cdots L}}_{m_1-2-l_1} \underbrace{\quad}_{l_1} \otimes \underbrace{\boxed{T \cdots T L \cdots L}}_{m_2+1-l_2} \underbrace{\quad}_{l_2} \bigotimes_{j \neq 1,2}^N \underbrace{\boxed{T \cdots T L \cdots L}}_{m_j-l_j} \underbrace{\quad}_{l_j} \\
& + l_1 \underbrace{\boxed{T \cdots T L \cdots L}}_{m_1-1-l_1} \underbrace{\quad}_{l_1-1} \otimes \underbrace{\boxed{T \cdots T L \cdots L}}_{m_2-l_2} \underbrace{\quad}_{l_2+1} \bigotimes_{j \neq 1,2}^N \underbrace{\boxed{T \cdots T L \cdots L}}_{m_j-l_j} \underbrace{\quad}_{l_j} \\
& + \sum_{l=3}^N (l-1) (m_{l-1} - l_{l-1}) \underbrace{\boxed{T \cdots T L \cdots L}}_{m_1-1-l_1} \underbrace{\quad}_{l_1} \otimes \underbrace{\boxed{T \cdots T}}_{m_{l-1}-1-l_{l-1}} \underbrace{\boxed{L \cdots L}}_{l_{l-1}} \\
& \otimes \underbrace{\boxed{T \cdots T L \cdots L}}_{m_l+1-l_l} \underbrace{\quad}_{l_l} \bigotimes_{j \neq 1,l,l-1}^N \underbrace{\boxed{T \cdots T L \cdots L}}_{m_j-l_j} \underbrace{\quad}_{l_j} \\
& + \sum_{l=3}^N l_{l-1} (l-1) \underbrace{\boxed{T \cdots T L \cdots L}}_{m_1-1-l_1} \underbrace{\quad}_{l_1} \otimes \underbrace{\boxed{T \cdots T L \cdots L}}_{m_{l-1}-l_{l-1}} \underbrace{\quad}_{l_{l-1}-1} \\
& \otimes \underbrace{\boxed{T \cdots T L \cdots L}}_{m_l-l_l} \underbrace{\quad}_{l_l+1} \bigotimes_{j \neq 1,l,l-1}^N \underbrace{\boxed{T \cdots T L \cdots L}}_{m_j-l_j} \underbrace{\quad}_{l_j},
\end{aligned} \tag{B.4a}$$

and

$$\begin{aligned}
0 = & \frac{1}{2} \underbrace{\overbrace{T \cdots T}^{m_1-l_1} \underbrace{L \cdots L}_{l_1}}_{j \neq 1} \overset{N}{\otimes} \underbrace{\overbrace{T \cdots T}^{m_j-l_j} \underbrace{L \cdots L}_{l_j}}_{j \neq 1} \\
& + M \underbrace{\overbrace{T \cdots T}^{m_1-2-l_1} \underbrace{L \cdots L}_{l_1}}_{j \neq 1,2} \overset{k}{\otimes} \underbrace{\overbrace{T \cdots T}^{m_2-l_2} \underbrace{L \cdots L}_{l_2+1}}_{j \neq 1,2} \overset{N}{\otimes} \underbrace{\overbrace{T \cdots T}^{m_j-l_j} \underbrace{L \cdots L}_{l_j}}_{j \neq 1,2} \\
& + (m_1 - 2 - l_1) \underbrace{\overbrace{T \cdots T}^{m_1-3-l_1} \underbrace{L \cdots L}_{l_1}}_{j \neq 1,3} \overset{N}{\otimes} \underbrace{\overbrace{T \cdots T}^{m_3+1-l_3} \underbrace{L \cdots L}_{l_3}}_{j \neq 1,3} \overset{N}{\otimes} \underbrace{\overbrace{T \cdots T}^{m_j-l_j} \underbrace{L \cdots L}_{l_j}}_{j \neq 1,3} \\
& + l_1 \underbrace{\overbrace{T \cdots T}^{m_1-2-l_1} \underbrace{L \cdots L}_{l_1-1}}_{j \neq 1,3} \overset{N}{\otimes} \underbrace{\overbrace{T \cdots T}^{m_3-l_3} \underbrace{L \cdots L}_{l_3+1}}_{j \neq 1,3} \overset{N}{\otimes} \underbrace{\overbrace{T \cdots T}^{m_j-l_j} \underbrace{L \cdots L}_{l_j}}_{j \neq 1,3} \\
& + \sum_{l=4}^N (l-2) (m_{l-2} - l_{l-2}) \underbrace{\overbrace{T \cdots T}^{m_1-2-l_1} \underbrace{L \cdots L}_{l_1}}_{j \neq 1, l-2} \overset{N}{\otimes} \underbrace{\overbrace{T \cdots T}^{m_{l-2}-1-l_{l-2}} \underbrace{L \cdots L}_{l_{l-2}}}_{j \neq 1, l-2} \\
& \otimes \underbrace{\overbrace{T \cdots T}^{m_l+1-l_l} \underbrace{L \cdots L}_{l_l}}_{j \neq 1, l-2} \overset{N}{\otimes} \underbrace{\overbrace{T \cdots T}^{m_j-l_j} \underbrace{L \cdots L}_{l_j}}_{j \neq 1, l-2} \\
& + \sum_{l=4}^N l_{l-2} (l-2) \underbrace{\overbrace{T \cdots T}^{m_1-2-l_1} \underbrace{L \cdots L}_{l_1}}_{j \neq 1, l-2} \overset{N}{\otimes} \sum_{i=2}^{m_{l-2}} \underbrace{\overbrace{T \cdots T}^{m_{l-2}-l_{l-2}} \underbrace{L \cdots L}_{l_{l-2}-1}}_{j \neq 1, l-2} \\
& \otimes \underbrace{\overbrace{T \cdots T}^{m_l-l_l} \underbrace{L \cdots L}_{l_l+1}}_{j \neq 1, l-2} \overset{N}{\otimes} \underbrace{\overbrace{T \cdots T}^{m_j-l_j} \underbrace{L \cdots L}_{l_j}}_{j \neq 1, l-2}. \tag{B.4b}
\end{aligned}$$

To solve the above constraints, we need the following lemma to further simplify them.

Lemma

$$\underbrace{\overbrace{T \cdots T}^{l_1} \underbrace{L \cdots L}_{l_1}}_{l_1} \otimes \underbrace{\overbrace{T \cdots T}^{m_2-l_2} \underbrace{L \cdots L}_{l_2}}_{m_2-l_2} \otimes \underbrace{\cdots}_{\{m_j, j \geq 3\}} \equiv 0, \tag{B.5}$$

except for (i) $l_2 = m_2$, $m_j = 0$ for $j \geq 3$ and (ii) $l_1 = 2m$.

Proof: In the high energy limit, we only need to consider the leading energy terms. To count the energy scaling behavior: each T contributes a factor of energy E and each L contributes E^2 . Any terms with total energy order level less than N are sub-leading terms and can be ignored.

(i) If $l_2 \neq m_2$ and $m_j \neq 0, j \geq 3$, then in Eq.(B.4a),

1. for $l_1 = 0$, all terms except the first term are sub-leading, then

$$\underbrace{\boxed{T \cdots T L \cdots L}}_1 \otimes_{j \neq 1}^N \underbrace{\boxed{T \cdots T L \cdots L}}_{m_j - l_j} \underbrace{\quad}_{l_j} = 0, \quad (\text{B.6})$$

2. for $l_1 = 1$, the third term is sub-leading, and (B.6) implies all other terms except the first term are vanished, then

$$\underbrace{\boxed{T \cdots T L \cdots L}}_2 \otimes_{j \neq 1}^N \underbrace{\boxed{T \cdots T L \cdots L}}_{m_j - l_j} \underbrace{\quad}_{l_j} = 0, \quad (\text{B.7})$$

3. if for $l_1 = l'$,

$$\underbrace{\boxed{T \cdots T L \cdots L}}_{l' - 1} \otimes_{j \neq 1}^N \underbrace{\boxed{T \cdots T L \cdots L}}_{m_j - l_j} \underbrace{\quad}_{l_j} = 0, \quad (\text{B.8})$$

and

$$\underbrace{\boxed{T \cdots T L \cdots L}}_{l'} \otimes_{j \neq 1}^N \underbrace{\boxed{T \cdots T L \cdots L}}_{m_j - l_j} \underbrace{\quad}_{l_j} = 0, \quad (\text{B.9})$$

(B.4a) implies all terms except the first term are vanished, then

$$\underbrace{\boxed{T \cdots T L \cdots L}}_{l' + 1} \otimes_{j \neq 1}^N \underbrace{\boxed{T \cdots T L \cdots L}}_{m_j - l_j} \underbrace{\quad}_{l_j} = 0. \quad (\text{B.10})$$

- (ii) If $l_2 = m_2$ and $m_j = 0$ for $j \geq 3$, then Eq.(B.4a) reduces to

$$M \underbrace{\boxed{T \cdots T L \cdots L}}_{m_1 - 1 - l_1} \underbrace{\quad}_{l_1 + 1} \otimes \underbrace{\boxed{L \cdots L}}_{m_2} + l_1 \underbrace{\boxed{T \cdots T L \cdots L}}_{m_1 - 1 - l_1} \underbrace{\quad}_{l_1 - 1} \otimes \underbrace{\boxed{L \cdots L}}_{m_2 + 1} = 0. \quad (\text{B.11})$$

Similarly, we have in Eq.(B.11),

1. for $l_1 = 0$,

$$\underbrace{\boxed{T \cdots T L \cdots L}}_1 \otimes \boxed{L \cdots L} = 0, \quad (\text{B.12})$$

2. if for $l_1 = 2m$,

$$\underbrace{\boxed{T \cdots T L \cdots L}}_{2m - 1} \otimes \boxed{L \cdots L} = 0, \quad (\text{B.13})$$

then (B.11) implies

$$\underbrace{\boxed{T \cdots T L \cdots L}}_{2m + 1} \otimes \boxed{L \cdots L} = 0. \quad (\text{B.14})$$

Finally, the Virasoro constraints at high energies reduce to

$$\underbrace{\boxed{T \cdots T}}_{n-2q-2-2m} \underbrace{\boxed{L \cdots L}}_{2m+2} \otimes \underbrace{\boxed{L \cdots L}}_q = -\frac{2m+1}{M} \underbrace{\boxed{T \cdots T}}_{n-2q-2-2m} \underbrace{\boxed{L \cdots L}}_{2m} \otimes \underbrace{\boxed{L \cdots L}}_{q+1}, \quad (\text{B.15a})$$

$$\underbrace{\boxed{T \cdots T}}_{n-2q-2-2m} \underbrace{\boxed{L \cdots L}}_{2m} \otimes \underbrace{\boxed{L \cdots L}}_{q+1} = -\frac{1}{2M} \underbrace{\boxed{T \cdots T}}_{n-2q-2m} \underbrace{\boxed{L \cdots L}}_{2m} \otimes \underbrace{\boxed{L \cdots L}}_q, \quad (\text{B.15b})$$

where we have renamed $m_2 \rightarrow q$ and $m_1 \rightarrow N - 2q$.

2. Superstring

Applying Virasoro conditions (8.27) and (8.28) on the states (8.39), we obtain

$$\begin{aligned} G_{1/2} |N\rangle = & \sum_{\{m_j, m_r\}} \left[k^{\nu_1^{1/2}} \underbrace{\boxed{\nu_1^{1/2} \cdots \nu_{m_{1/2}}^{1/2}}}_T \underbrace{\bigotimes_{j=1}^N \boxed{\mu_1^j \cdots \mu_{m_j}^j}}_{j \neq l} \underbrace{\bigotimes_{r \neq 1/2}^{N-1/2} \boxed{\nu_1^r \cdots \nu_{m_r}^r}}_T \right. \\ & + \sum_{l \geq 1} \sum_{i=1}^{m_l} l \underbrace{\boxed{\mu_1^l \cdots \hat{\mu}_i^l \cdots \mu_{m_l}^l}} \otimes \underbrace{\boxed{\mu_i^l \nu_1^{l+1/2} \cdots \nu_{m_{l+1/2}}^{l+1/2}}}_T \\ & \otimes \underbrace{\boxed{\nu_2^{1/2} \cdots \nu_{m_{1/2}}^{1/2}}}_T \underbrace{\bigotimes_{j \neq l}^N \boxed{\mu_1^j \cdots \mu_{m_j}^j}}_{j \neq l} \underbrace{\bigotimes_{r \neq 1/2, l+1/2}^{N-1/2} \boxed{\nu_1^r \cdots \nu_{m_r}^r}}_T \\ & + \sum_{i=2}^{m_{1/2}} \underbrace{\boxed{\nu_i^{1/2} \mu_1^1 \cdots \mu_{m_1}^1}} \otimes (-1)^{i+1} \underbrace{\boxed{\nu_2^{1/2} \cdots \hat{\nu}_i^{1/2} \cdots \nu_{m_{1/2}}^{1/2}}}_T \\ & \underbrace{\bigotimes_{j \neq l}^N \boxed{\mu_1^j \cdots \mu_{m_j}^j}}_{j \neq l} \underbrace{\bigotimes_{r \neq 1/2}^{N-1/2} \boxed{\nu_1^r \cdots \nu_{m_r}^r}}_T \\ & + \sum_{l \geq 2} \sum_{i=1}^{m_{l-1/2}} (-1)^{i+1} \underbrace{\boxed{\nu_i^{l-1/2} \mu_1^l \cdots \mu_{m_l}^l}} \otimes \underbrace{\boxed{\nu_1^{l-1/2} \cdots \hat{\nu}_i^{l-1/2} \cdots \nu_{m_{l-1/2}}^{l-1/2}}}_T \\ & \otimes \underbrace{\boxed{\nu_2^{1/2} \cdots \nu_{m_{1/2}}^{1/2}}}_T \underbrace{\bigotimes_{j \neq l}^N \boxed{\mu_1^j \cdots \mu_{m_j}^j}}_{j \neq l} \underbrace{\bigotimes_{r \neq 1/2, l-1/2}^{N-1/2} \boxed{\nu_1^r \cdots \nu_{m_r}^r}}_T \Big] \\ & \cdot \frac{1}{(m_{1/2} - 1)!} \psi_{-1/2}^{\nu_2^{1/2} \cdots \nu_{m_{1/2}}^{1/2}} \prod_{j=1}^N \frac{1}{j^{m_j} m_j!} \alpha_{-j}^{\mu_1^j \cdots \mu_{m_j}^j} \prod_{r \neq 1/2}^{N-1/2} \frac{1}{m_r!} \psi_{-r}^{\nu_1^r \cdots \nu_{m_r}^r}, \quad (\text{B.16a}) \end{aligned}$$

and

$$\begin{aligned}
G_{3/2} |N\rangle = & \sum_{\{m_j, m_r\}} \left[\left[\nu_1^{3/2} \dots \nu_{m_{3/2}}^{3/2} \right]^T k^{\nu_1^{3/2}} \bigotimes_{j=1}^N \left[\mu_1^j \dots \mu_{m_j}^j \right] \bigotimes_{r \neq 3/2}^{N-1/2} \left[\nu_1^r \dots \nu_{m_r}^r \right]^T \right. \\
& + \eta^{\mu\nu} \left[\mu \mu_1^1 \dots \mu_{m_1}^1 \right] \otimes \left[\nu \nu_1^{1/2} \dots \nu_{m_{1/2}}^{1/2} \right]^T \otimes \left[\nu_2^{3/2} \dots \nu_{m_{3/2}}^{3/2} \right]^T \\
& \bigotimes_{j \neq 1}^N \left[\mu_1^j \dots \mu_{m_j}^j \right] \bigotimes_{r \neq 1/2, 3/2}^{N-1/2} \left[\nu_1^r \dots \nu_{m_r}^r \right]^T \\
& + \sum_{l \geq 1} \sum_{i=1}^{m_l} l \left[\mu_1^l \dots \hat{\mu}_i^l \dots \mu_{m_l}^l \right] \otimes \left[\mu_i^l \nu_1^{l+3/2} \dots \nu_{m_{l+3/2}}^{l+3/2} \right]^T \\
& \otimes \left[\nu_2^{3/2} \dots \nu_{m_{3/2}}^{3/2} \right]^T \bigotimes_{j \neq l}^N \left[\mu_1^j \dots \mu_{m_j}^j \right] \bigotimes_{r \neq 3/2, l+3/2}^{N-1/2} \left[\nu_1^r \dots \nu_{m_r}^r \right]^T \\
& + \sum_{i=2}^{m_{3/2}} 3 \left[\nu_i^{3/2} \mu_1^3 \dots \mu_{m_3}^3 \right] \otimes (-1)^{i+1} \left[\nu_2^{3/2} \dots \hat{\nu}_i^{3/2} \dots \nu_{m_{3/2}}^{3/2} \right]^T \\
& \bigotimes_{j \neq 3}^N \left[\mu_1^j \dots \mu_{m_j}^j \right] \bigotimes_{r \neq 3/2}^{N-1/2} \left[\nu_1^r \dots \nu_{m_r}^r \right]^T \\
& + \sum_{l \geq 2, l \neq 3} \sum_{i=1}^{m_{l-3/2}} \left[\nu_i^{l-3/2} \mu_1^l \dots \mu_{m_l}^l \right] \otimes \left[\nu_1^{l-3/2} \dots \hat{\nu}_i^{l-3/2} \dots \nu_{m_{l-3/2}}^{l-3/2} \right]^T \\
& \otimes \left[\nu_2^{3/2} \dots \nu_{m_{3/2}}^{3/2} \right]^T \bigotimes_{j \neq l}^N \left[\mu_1^j \dots \mu_{m_j}^j \right] \bigotimes_{r \neq 3/2, l-3/2}^{N-1/2} \left[\nu_1^r \dots \nu_{m_r}^r \right]^T \Big] \\
& \cdot \frac{1}{(m_{3/2} - 1)!} \psi_{-3/2}^{\nu_2^{3/2} \dots \nu_{m_{3/2}}^{3/2}} \prod_{j=1}^N \frac{1}{j^{m_j} m_j!} \alpha_{-j}^{\mu_1^j \dots \mu_{m_j}^j} \prod_{r \neq 3/2}^{N-1/2} \frac{1}{m_r!} \psi_{-r}^{\nu_1^r \dots \nu_{m_r}^r}, \tag{B.16b}
\end{aligned}$$

where we have used the identities of the Young tableaux,

$$\begin{aligned}
\left[1 \dots p \right] &= \frac{1}{p} \left[1 + \sigma_{(21)} + \sigma_{(321)} + \dots + \sigma_{(p \dots 1)} \right] \left[1 \right] \otimes \left[2 \dots p \right] \\
&= \frac{1}{p} \sum_{i=1}^p \sigma_{(i1)} \left[1 \right] \otimes \left[2 \dots p \right], \tag{B.17}
\end{aligned}$$

$$\begin{aligned}
\left[1 \dots p \right]^T &= \frac{1}{p} \left[1 - \sigma_{(21)} + \sigma_{(321)} - \dots + (-1)^{p+1} \sigma_{(p \dots 1)} \right] \left[1 \right] \otimes \left[2 \dots p \right]^T \\
&= \frac{1}{p} \sum_{i=1}^p (-1)^{i+1} \sigma_{(i \dots 1)} \left[1 \right] \otimes \left[2 \dots p \right]^T. \tag{B.18}
\end{aligned}$$

We then obtain the constraint equations

$$\begin{aligned}
0 = & k^{\nu_1^{1/2}} \left[\nu_1^{1/2} \cdots \nu_{m_1/2}^{1/2} \right]^T \bigotimes_{j=1}^N \left[\mu_1^j \cdots \mu_{m_j}^j \right] \bigotimes_{r \neq 1/2}^{N-1/2} \left[\nu_1^r \cdots \nu_{m_r}^r \right]^T \\
& + \sum_{l \geq 1} \sum_{i=1}^{m_l} l \left[\mu_1^l \cdots \hat{\mu}_i^l \cdots \mu_{m_l}^l \right] \otimes \left[\mu_i^l \nu_1^{l+1/2} \cdots \nu_{m_{l+1/2}}^{l+1/2} \right]^T \\
& \otimes \left[\nu_2^{1/2} \cdots \nu_{m_1/2}^{1/2} \right]^T \bigotimes_{j \neq l}^N \left[\mu_1^j \cdots \mu_{m_j}^j \right] \bigotimes_{r \neq 1/2, l+1/2}^{N-1/2} \left[\nu_1^r \cdots \nu_{m_r}^r \right]^T \\
& + \sum_{i=2}^{m_1/2} \left[\nu_i^{1/2} \mu_1^1 \cdots \mu_{m_1}^1 \right] \otimes (-1)^{i+1} \left[\nu_2^{1/2} \cdots \hat{\nu}_i^{1/2} \cdots \nu_{m_1/2}^{1/2} \right]^T \\
& \bigotimes_{j \neq l}^N \left[\mu_1^j \cdots \mu_{m_j}^j \right] \bigotimes_{r \neq 1/2}^{N-1/2} \left[\nu_1^r \cdots \nu_{m_r}^r \right]^T \\
& + \sum_{l \geq 2} \sum_{i=1}^{m_{l-1/2}} (-1)^{i+1} \left[\nu_i^{l-1/2} \mu_1^l \cdots \mu_{m_l}^l \right] \otimes \left[\nu_1^{l-1/2} \cdots \hat{\nu}_i^{l-1/2} \cdots \nu_{m_{l-1/2}}^{l-1/2} \right]^T \\
& \otimes \left[\nu_2^{1/2} \cdots \nu_{m_1/2}^{1/2} \right]^T \bigotimes_{j \neq l}^N \left[\mu_1^j \cdots \mu_{m_j}^j \right] \bigotimes_{r \neq 1/2, l-1/2}^{N-1/2} \left[\nu_1^r \cdots \nu_{m_r}^r \right]^T, \tag{B.19a}
\end{aligned}$$

$$\begin{aligned}
0 = & \left[\nu_1^{3/2} \cdots \nu_{m_3/2}^{3/2} \right]^T k^{\nu_1^{3/2}} \bigotimes_{j=1}^N \left[\mu_1^j \cdots \mu_{m_j}^j \right] \bigotimes_{r \neq 3/2}^{N-1/2} \left[\nu_1^r \cdots \nu_{m_r}^r \right]^T \\
& + \eta^{\mu\nu} \left[\mu \mu_1^1 \cdots \mu_{m_1}^1 \right] \otimes \left[\nu \nu_1^{1/2} \cdots \nu_{m_1/2}^{1/2} \right]^T \otimes \left[\nu_2^{3/2} \cdots \nu_{m_3/2}^{3/2} \right]^T \\
& \bigotimes_{j \neq 1}^N \left[\mu_1^j \cdots \mu_{m_j}^j \right] \bigotimes_{r \neq 1/2, 3/2}^{N-1/2} \left[\nu_1^r \cdots \nu_{m_r}^r \right]^T \\
& + \sum_{l \geq 1} \sum_{i=1}^{m_l} l \left[\mu_1^l \cdots \hat{\mu}_i^l \cdots \mu_{m_l}^l \right] \otimes \left[\mu_i^l \nu_1^{l+3/2} \cdots \nu_{m_{l+3/2}}^{l+3/2} \right]^T \\
& \otimes \left[\nu_2^{3/2} \cdots \nu_{m_3/2}^{3/2} \right]^T \bigotimes_{j \neq l}^N \left[\mu_1^j \cdots \mu_{m_j}^j \right] \bigotimes_{r \neq 3/2, l+3/2}^{N-1/2} \left[\nu_1^r \cdots \nu_{m_r}^r \right]^T \\
& + \sum_{i=2}^{m_3/2} 3 \left[\nu_i^{3/2} \mu_1^3 \cdots \mu_{m_3}^3 \right] \otimes (-1)^{i+1} \left[\nu_2^{3/2} \cdots \hat{\nu}_i^{3/2} \cdots \nu_{m_3/2}^{3/2} \right]^T \\
& \bigotimes_{j \neq 3}^N \left[\mu_1^j \cdots \mu_{m_j}^j \right] \bigotimes_{r \neq 3/2}^{N-1/2} \left[\nu_1^r \cdots \nu_{m_r}^r \right]^T \\
& + \sum_{l \geq 2, l \neq 3} \sum_{i=1}^{m_{l-3/2}} \left[\nu_i^{l-3/2} \mu_1^l \cdots \mu_{m_l}^l \right] \otimes (-1)^{i+1} \left[\nu_1^{l-3/2} \cdots \hat{\nu}_i^{l-3/2} \cdots \nu_{m_{l-3/2}}^{l-3/2} \right]^T \\
& \otimes \left[\nu_2^{3/2} \cdots \nu_{m_3/2}^{3/2} \right]^T \bigotimes_{j \neq l}^N \left[\mu_1^j \cdots \mu_{m_j}^j \right] \bigotimes_{r \neq 3/2, l-3/2}^{N-1/2} \left[\nu_1^r \cdots \nu_{m_r}^r \right]^T. \tag{B.19b}
\end{aligned}$$

Taking the high energy limit in the above equations by letting $(\mu_i, \nu_i) \rightarrow (L, T)$, and

$$k^{\mu_i} \rightarrow M (e^L)^{\mu_i}, \eta^{\mu_1 \mu_2} \rightarrow (e^T)^{\mu_1} (e^T)^{\mu_2}, \tag{B.20}$$

we get

$$\begin{aligned}
0 = & M \left[L \begin{array}{|c|c|c|} \hline \nu_2^{1/2} & \dots & \nu_{m_{1/2}}^{1/2} \\ \hline \end{array} \right]^T \bigotimes_{j=1}^N \left[\begin{array}{|c|c|c|} \hline \mu_1^j & \dots & \mu_{m_j}^j \\ \hline \end{array} \right] \bigotimes_{r \neq 1/2}^{N-1/2} \left[\begin{array}{|c|c|c|} \hline \nu_1^r & \dots & \nu_{m_r}^r \\ \hline \end{array} \right]^T \\
& + \sum_{l \geq 1} \sum_{i=1}^{m_l} l \left[\begin{array}{|c|c|c|c|} \hline \mu_1^l & \dots & \hat{\mu}_i^l & \dots & \mu_{m_l}^l \\ \hline \end{array} \right] \otimes \left[\begin{array}{|c|c|c|c|} \hline \mu_i^l & \nu_1^{l+1/2} & \dots & \nu_{m_{l+1/2}}^{l+1/2} \\ \hline \end{array} \right]^T \\
& \otimes \left[\begin{array}{|c|c|c|} \hline \nu_2^{1/2} & \dots & \nu_{m_{1/2}}^{1/2} \\ \hline \end{array} \right]^T \bigotimes_{j \neq l}^N \left[\begin{array}{|c|c|c|} \hline \mu_1^j & \dots & \mu_{m_j}^j \\ \hline \end{array} \right] \bigotimes_{r \neq 1/2, l+1/2}^{N-1/2} \left[\begin{array}{|c|c|c|} \hline \nu_1^r & \dots & \nu_{m_r}^r \\ \hline \end{array} \right]^T \\
& + \sum_{i=2}^{m_{1/2}} \left[\begin{array}{|c|c|c|c|} \hline \nu_i^{1/2} & \mu_1^1 & \dots & \mu_{m_1}^1 \\ \hline \end{array} \right] \otimes (-1)^{i+1} \left[\begin{array}{|c|c|c|c|} \hline \nu_2^{1/2} & \dots & \hat{\nu}_i^{1/2} & \dots & \nu_{m_{1/2}}^{1/2} \\ \hline \end{array} \right]^T \\
& \bigotimes_{j \neq l}^N \left[\begin{array}{|c|c|c|} \hline \mu_1^j & \dots & \mu_{m_j}^j \\ \hline \end{array} \right] \bigotimes_{r \neq 1/2}^{N-1/2} \left[\begin{array}{|c|c|c|} \hline \nu_1^r & \dots & \nu_{m_r}^r \\ \hline \end{array} \right]^T \\
& + \sum_{l \geq 2} \sum_{i=1}^{m_{l-1/2}} (-1)^{i+1} \left[\begin{array}{|c|c|c|c|} \hline \nu_i^{l-1/2} & \mu_1^l & \dots & \mu_{m_l}^l \\ \hline \end{array} \right] \otimes \left[\begin{array}{|c|c|c|c|} \hline \nu_1^{l-1/2} & \dots & \hat{\nu}_i^{l-1/2} & \dots & \nu_{m_{l-1/2}}^{l-1/2} \\ \hline \end{array} \right]^T \\
& \otimes \left[\begin{array}{|c|c|c|} \hline \nu_2^{1/2} & \dots & \nu_{m_{1/2}}^{1/2} \\ \hline \end{array} \right]^T \bigotimes_{j \neq l}^N \left[\begin{array}{|c|c|c|} \hline \mu_1^j & \dots & \mu_{m_j}^j \\ \hline \end{array} \right] \bigotimes_{r \neq 1/2, l-1/2}^{N-1/2} \left[\begin{array}{|c|c|c|} \hline \nu_1^r & \dots & \nu_{m_r}^r \\ \hline \end{array} \right]^T, \tag{B.21a}
\end{aligned}$$

$$\begin{aligned}
0 = & M \left[L \begin{array}{|c|c|c|} \hline \nu_2^{3/2} & \dots & \nu_{m_{3/2}}^{3/2} \\ \hline \end{array} \right]^T \bigotimes_{j=1}^N \left[\begin{array}{|c|c|c|} \hline \mu_1^j & \dots & \mu_{m_j}^j \\ \hline \end{array} \right] \bigotimes_{r \neq 3/2}^{N-1/2} \left[\begin{array}{|c|c|c|} \hline \nu_1^r & \dots & \nu_{m_r}^r \\ \hline \end{array} \right]^T \\
& + \left[T \begin{array}{|c|c|c|} \hline \mu_1^1 & \dots & \mu_{m_1}^1 \\ \hline \end{array} \right] \otimes \left[T \begin{array}{|c|c|c|} \hline \nu_1^{1/2} & \dots & \nu_{m_{1/2}}^{1/2} \\ \hline \end{array} \right]^T \otimes \left[\begin{array}{|c|c|c|} \hline \nu_2^{3/2} & \dots & \nu_{m_{3/2}}^{3/2} \\ \hline \end{array} \right]^T \\
& \bigotimes_{j \neq 1}^N \left[\begin{array}{|c|c|c|} \hline \mu_1^j & \dots & \mu_{m_j}^j \\ \hline \end{array} \right] \bigotimes_{r \neq 1/2, 3/2}^{N-1/2} \left[\begin{array}{|c|c|c|} \hline \nu_1^r & \dots & \nu_{m_r}^r \\ \hline \end{array} \right]^T \\
& + \sum_{l \geq 1} \sum_{i=1}^{m_l} l \left[\begin{array}{|c|c|c|c|} \hline \mu_1^l & \dots & \hat{\mu}_i^l & \dots & \mu_{m_l}^l \\ \hline \end{array} \right] \otimes \left[\begin{array}{|c|c|c|c|} \hline \mu_i^l & \nu_1^{l+3/2} & \dots & \nu_{m_{l+3/2}}^{l+3/2} \\ \hline \end{array} \right]^T \\
& \otimes \left[\begin{array}{|c|c|c|} \hline \nu_2^{3/2} & \dots & \nu_{m_{3/2}}^{3/2} \\ \hline \end{array} \right]^T \bigotimes_{j \neq l}^N \left[\begin{array}{|c|c|c|} \hline \mu_1^j & \dots & \mu_{m_j}^j \\ \hline \end{array} \right] \bigotimes_{r \neq 3/2, l+3/2}^{N-1/2} \left[\begin{array}{|c|c|c|} \hline \nu_1^r & \dots & \nu_{m_r}^r \\ \hline \end{array} \right]^T \\
& + \sum_{i=2}^{m_{3/2}} 3 \left[\begin{array}{|c|c|c|c|} \hline \nu_i^{3/2} & \mu_1^3 & \dots & \mu_{m_3}^3 \\ \hline \end{array} \right] \otimes (-1)^{i+1} \left[\begin{array}{|c|c|c|c|} \hline \nu_2^{3/2} & \dots & \hat{\nu}_i^{3/2} & \dots & \nu_{m_{3/2}}^{3/2} \\ \hline \end{array} \right]^T \\
& \bigotimes_{j \neq 3}^N \left[\begin{array}{|c|c|c|} \hline \mu_1^j & \dots & \mu_{m_j}^j \\ \hline \end{array} \right] \bigotimes_{r \neq 3/2}^{N-1/2} \left[\begin{array}{|c|c|c|} \hline \nu_1^r & \dots & \nu_{m_r}^r \\ \hline \end{array} \right]^T \\
& + \sum_{l \geq 2, l \neq 3} \sum_{i=1}^{m_{l-3/2}} \left[\begin{array}{|c|c|c|c|} \hline \nu_i^{l-3/2} & \mu_1^l & \dots & \mu_{m_l}^l \\ \hline \end{array} \right] \otimes (-1)^{i+1} \left[\begin{array}{|c|c|c|c|} \hline \nu_1^{l-3/2} & \dots & \hat{\nu}_i^{l-3/2} & \dots & \nu_{m_{l-3/2}}^{l-3/2} \\ \hline \end{array} \right]^T \\
& \otimes \left[\begin{array}{|c|c|c|} \hline \nu_2^{3/2} & \dots & \nu_{m_{3/2}}^{3/2} \\ \hline \end{array} \right]^T \bigotimes_{j \neq l}^N \left[\begin{array}{|c|c|c|} \hline \mu_1^j & \dots & \mu_{m_j}^j \\ \hline \end{array} \right] \bigotimes_{r \neq 3/2, l-3/2}^{N-1/2} \left[\begin{array}{|c|c|c|} \hline \nu_1^r & \dots & \nu_{m_r}^r \\ \hline \end{array} \right]^T. \tag{B.21b}
\end{aligned}$$

The indices $\{\mu_i^j\}$ are symmetric and can be chosen to have l_j of $\{L\}$ and $\{T\}$, while $\{\nu_i^r\}$ are antisymmetric and we keep them as what they are at this moment. Thus

$$\begin{aligned}
0 = & M \left[L \nu_2^{1/2} \cdots \nu_{m_1/2}^{1/2} \right]^T \bigotimes_{j=1}^N \underbrace{\left[\mu_1^j T \cdots T L \cdots L \right]}_{m_j-1-l_j} \bigotimes_{r \neq 1/2}^{N-1/2} \left[\nu_1^r \cdots \nu_{m_r}^r \right]^T \\
& + \sum_{l \geq 1} l \underbrace{\left[T \cdots T L \cdots L \right]}_{m_l-1-l_l} \otimes_{l_l} \left[\mu_1^l \nu_1^{l+1/2} \cdots \nu_{m_{l+1/2}}^{l+1/2} \right]^T \\
& \otimes \left[\nu_2^{1/2} \cdots \nu_{m_1/2}^{1/2} \right]^T \bigotimes_{j \neq l}^N \underbrace{\left[\mu_1^j T \cdots T L \cdots L \right]}_{m_j-1-l_j} \bigotimes_{r \neq 1/2, l+1/2}^{N-1/2} \left[\nu_1^r \cdots \nu_{m_r}^r \right]^T \\
& + \sum_{l \geq 1} l(m_l - 1 - l_l) \underbrace{\left[\mu_1^l T \cdots T L \cdots L \right]}_{m_l-2-l_l} \otimes_{l_l} \left[T \nu_1^{l+1/2} \cdots \nu_{m_{l+1/2}}^{l+1/2} \right]^T \\
& \otimes \left[\nu_2^{1/2} \cdots \nu_{m_1/2}^{1/2} \right]^T \bigotimes_{j \neq l}^N \underbrace{\left[\mu_1^j T \cdots T L \cdots L \right]}_{m_j-1-l_j} \bigotimes_{r \neq 1/2, l+1/2}^{N-1/2} \left[\nu_1^r \cdots \nu_{m_r}^r \right]^T \\
& + \sum_{l \geq 1} l l_l \underbrace{\left[\mu_1^l T \cdots T L \cdots L \right]}_{m_l-1-l_l} \otimes_{l_l-1} \left[L \nu_1^{l+1/2} \cdots \nu_{m_{l+1/2}}^{l+1/2} \right]^T \\
& \otimes \left[\nu_2^{1/2} \cdots \nu_{m_1/2}^{1/2} \right]^T \bigotimes_{j \neq l}^N \underbrace{\left[\mu_1^j T \cdots T L \cdots L \right]}_{m_j-1-l_j} \bigotimes_{r \neq 1/2, l+1/2}^{N-1/2} \left[\nu_1^r \cdots \nu_{m_r}^r \right]^T \\
& + \sum_{i=2}^{m_1/2} \underbrace{\left[\nu_i^{1/2} \mu_1^1 T \cdots T L \cdots L \right]}_{m_1-1-l_1} \otimes_{l_1} (-1)^{i+1} \left[\nu_2^{1/2} \cdots \hat{\nu}_i^{1/2} \cdots \nu_{m_1/2}^{1/2} \right]^T \\
& \bigotimes_{j \neq 1}^N \underbrace{\left[\mu_1^j T \cdots T L \cdots L \right]}_{m_j-1-l_j} \bigotimes_{r \neq 1/2}^{N-1/2} \left[\nu_1^r \cdots \nu_{m_r}^r \right]^T \\
& + \sum_{l \geq 2} \sum_{i=1}^{m_{l-1/2}} (-1)^{i+1} \underbrace{\left[\nu_i^{l-1/2} \mu_1^l T \cdots T L \cdots L \right]}_{m_l-1-l_l} \otimes_{l_l} \left[\nu_1^{l-1/2} \cdots \hat{\nu}_i^{l-1/2} \cdots \nu_{m_{l-1/2}}^{l-1/2} \right]^T \\
& \otimes \left[\nu_2^{1/2} \cdots \nu_{m_1/2}^{1/2} \right]^T \bigotimes_{j \neq l}^N \underbrace{\left[\mu_1^j T \cdots T L \cdots L \right]}_{m_j-1-l_j} \bigotimes_{r \neq 1/2, l-1/2}^{N-1/2} \left[\nu_1^r \cdots \nu_{m_r}^r \right]^T, \tag{B.22a}
\end{aligned}$$

$$\begin{aligned}
0 = & M \left[L \nu_2^{3/2} \cdots \nu_{m_{3/2}}^{3/2} \right]^T \bigotimes_{j=1}^N \underbrace{\left[\mu_1^j T \cdots T L \cdots L \right]}_{m_j-1-l_j} \bigotimes_{r \neq 3/2}^{N-1/2} \left[\nu_1^r \cdots \nu_{m_r}^r \right]^T \\
& + \underbrace{\left[\mu_1^1 T \cdots T L \cdots L \right]}_{m_1-l_1} \otimes \underbrace{\left[T \nu_1^{1/2} \cdots \nu_{m_{1/2}}^{1/2} \right]}_{l_1}^T \otimes \left[\nu_2^{3/2} \cdots \nu_{m_{3/2}}^{3/2} \right]^T \\
& \bigotimes_{j \neq 1}^N \underbrace{\left[\mu_1^j T \cdots T L \cdots L \right]}_{m_j-1-l_j} \bigotimes_{r \neq 1/2, 3/2}^{N-1/2} \left[\nu_1^r \cdots \nu_{m_r}^r \right]^T \\
& + \sum_{l \geq 1} l \underbrace{\left[T \cdots T L \cdots L \right]}_{m_l-1-l_l} \otimes \underbrace{\left[\mu_1^l \nu_1^{l+3/2} \cdots \nu_{m_{l+3/2}}^{l+3/2} \right]}_{l_l}^T \\
& \otimes \left[\nu_2^{3/2} \cdots \nu_{m_{3/2}}^{3/2} \right]^T \bigotimes_{j \neq l}^N \underbrace{\left[\mu_1^j T \cdots T L \cdots L \right]}_{m_j-1-l_j} \bigotimes_{r \neq 3/2, l+3/2}^{N-1/2} \left[\nu_1^r \cdots \nu_{m_r}^r \right]^T \\
& + \sum_{l \geq 1} l (m_l - 1 - l_l) \underbrace{\left[\mu_1^l T \cdots T L \cdots L \right]}_{m_l-2-l_l} \otimes \underbrace{\left[T \nu_1^{l+3/2} \cdots \nu_{m_{l+3/2}}^{l+3/2} \right]}_{l_l}^T \\
& \otimes \left[\nu_2^{3/2} \cdots \nu_{m_{3/2}}^{3/2} \right]^T \bigotimes_{j \neq l}^N \underbrace{\left[\mu_1^j T \cdots T L \cdots L \right]}_{m_j-1-l_j} \bigotimes_{r \neq 3/2, l+3/2}^{N-1/2} \left[\nu_1^r \cdots \nu_{m_r}^r \right]^T \\
& + \sum_{l \geq 1} l l_l \underbrace{\left[\mu_1^l T \cdots T L \cdots L \right]}_{m_l-1-l_l} \otimes \underbrace{\left[L \nu_1^{l+3/2} \cdots \nu_{m_{l+3/2}}^{l+3/2} \right]}_{l_l-1}^T \\
& \otimes \left[\nu_2^{3/2} \cdots \nu_{m_{3/2}}^{3/2} \right]^T \bigotimes_{j \neq l}^N \underbrace{\left[\mu_1^j T \cdots T L \cdots L \right]}_{m_j-1-l_j} \bigotimes_{r \neq 3/2, l+3/2}^{N-1/2} \left[\nu_1^r \cdots \nu_{m_r}^r \right]^T \\
& + \sum_{i=2}^{m_{3/2}} 3 \underbrace{\left[\nu_i^{3/2} \mu_1^3 T \cdots T L \cdots L \right]}_{m_3-1-l_3} \otimes (-1)^{i+1} \left[\nu_2^{3/2} \cdots \hat{\nu}_i^{3/2} \cdots \nu_{m_{3/2}}^{3/2} \right]^T \\
& \bigotimes_{j \neq 3}^N \underbrace{\left[\mu_1^j T \cdots T L \cdots L \right]}_{m_j-1-l_j} \bigotimes_{r \neq 3/2}^{N-1/2} \left[\nu_1^r \cdots \nu_{m_r}^r \right]^T \\
& + \sum_{l \geq 2, l \neq 3} \sum_{i=1}^{m_{l-3/2}} \underbrace{\left[\nu_i^{l-3/2} \mu_1^l T \cdots T L \cdots L \right]}_{m_l-1-l_l} \otimes (-1)^{i+1} \left[\nu_1^{l-3/2} \cdots \hat{\nu}_i^{l-3/2} \cdots \nu_{m_{l-3/2}}^{l-3/2} \right]^T \\
& \otimes \left[\nu_2^{3/2} \cdots \nu_{m_{3/2}}^{3/2} \right]^T \bigotimes_{j \neq l}^N \underbrace{\left[\mu_1^j T \cdots T L \cdots L \right]}_{m_j-1-l_j} \bigotimes_{r \neq 3/2, l-3/2}^{N-1/2} \left[\nu_1^r \cdots \nu_{m_r}^r \right]^T. \tag{B.22b}
\end{aligned}$$

There are some undetermined parameters, which can be L or T , in the above equations. However, it is easy to see that both choice lead to the same equations. Therefore, we will

set all of them to be T in the following. Thus, the constrain equations become

$$\begin{aligned}
0 = & M \left[L \nu_2^{1/2} \cdots \nu_{m_1/2}^{1/2} \right]^T \bigotimes_{j=1}^N \underbrace{\left[T \cdots T L \cdots L \right]}_{m_j-l_j} \bigotimes_{r \neq 1/2}^{N-1/2} \left[\nu_1^r \cdots \nu_{m_r}^r \right]^T \\
& + \sum_{l \geq 1} l (m_l - l_l) \underbrace{\left[T \cdots T L \cdots L \right]}_{m_l-1-l_l} \bigotimes_{l_l} \left[T \nu_1^{l+1/2} \cdots \nu_{m_{l+1/2}}^{l+1/2} \right]^T \\
& \otimes \left[\nu_2^{1/2} \cdots \nu_{m_1/2}^{1/2} \right]^T \bigotimes_{j \neq l}^N \underbrace{\left[T \cdots T L \cdots L \right]}_{m_j-l_j} \bigotimes_{r \neq 1/2, l+1/2}^{N-1/2} \left[\nu_1^r \cdots \nu_{m_r}^r \right]^T \\
& + \sum_{l \geq 1} l l_l \underbrace{\left[T \cdots T L \cdots L \right]}_{m_l-l_l} \bigotimes_{l_l-1} \left[L \nu_1^{l+1/2} \cdots \nu_{m_{l+1/2}}^{l+1/2} \right]^T \\
& \otimes \left[\nu_2^{1/2} \cdots \nu_{m_1/2}^{1/2} \right]^T \bigotimes_{j \neq l}^N \underbrace{\left[T \cdots T L \cdots L \right]}_{m_j-l_j} \bigotimes_{r \neq 1/2, l+1/2}^{N-1/2} \left[\nu_1^r \cdots \nu_{m_r}^r \right]^T \\
& + \sum_{i=2}^{m_1/2} \underbrace{\left[\nu_i^{1/2} T \cdots T L \cdots L \right]}_{m_1-l_1} \bigotimes_{l_1} (-1)^{i+1} \left[\nu_2^{1/2} \cdots \hat{\nu}_i^{1/2} \cdots \nu_{m_1/2}^{1/2} \right]^T \\
& \bigotimes_{j \neq 1}^N \underbrace{\left[T \cdots T L \cdots L \right]}_{m_j-l_j} \bigotimes_{r \neq 1/2}^{N-1/2} \left[\nu_1^r \cdots \nu_{m_r}^r \right]^T \\
& + \sum_{l \geq 2} \sum_{i=1}^{m_{l-1}/2} (-1)^{i+1} \underbrace{\left[\nu_i^{l-1/2} T \cdots T L \cdots L \right]}_{m_l-l_l} \bigotimes_{l_l} \left[\nu_1^{l-1/2} \cdots \hat{\nu}_i^{l-1/2} \cdots \nu_{m_{l-1/2}}^{l-1/2} \right]^T \\
& \otimes \left[\nu_2^{1/2} \cdots \nu_{m_1/2}^{1/2} \right]^T \bigotimes_{j \neq l}^N \underbrace{\left[T \cdots T L \cdots L \right]}_{m_j-l_j} \bigotimes_{r \neq 1/2, l-1/2}^{N-1/2} \left[\nu_1^r \cdots \nu_{m_r}^r \right]^T, \tag{B.23a}
\end{aligned}$$

$$\begin{aligned}
0 = & M \left[\boxed{L} \boxed{\nu_2^{3/2}} \cdots \boxed{\nu_{m_{3/2}}^{3/2}} \right]^T \bigotimes_{j=1}^N \underbrace{\boxed{T} \cdots \boxed{T}}_{m_j-l_j} \underbrace{\boxed{L} \cdots \boxed{L}}_{l_j} \bigotimes_{r \neq 3/2}^{N-1/2} \boxed{\nu_1^r} \cdots \boxed{\nu_{m_r}^r}^T \\
& + \underbrace{\boxed{T} \cdots \boxed{T}}_{m_1+1-l_1} \underbrace{\boxed{L} \cdots \boxed{L}}_{l_1} \otimes \boxed{T} \boxed{\nu_1^{1/2}} \cdots \boxed{\nu_{m_{1/2}}^{1/2}}^T \otimes \boxed{\nu_2^{3/2}} \cdots \boxed{\nu_{m_{3/2}}^{3/2}}^T \\
& \bigotimes_{j \neq 1}^N \underbrace{\boxed{T} \cdots \boxed{T}}_{m_j-l_j} \underbrace{\boxed{L} \cdots \boxed{L}}_{l_j} \bigotimes_{r \neq 1/2, 3/2}^{N-1/2} \boxed{\nu_1^r} \cdots \boxed{\nu_{m_r}^r}^T \\
& + \sum_{l \geq 1} l (m_l - l_l) \underbrace{\boxed{T} \cdots \boxed{T}}_{m_l-1-l_l} \underbrace{\boxed{L} \cdots \boxed{L}}_{l_l} \otimes \boxed{T} \boxed{\nu_1^{l+3/2}} \cdots \boxed{\nu_{m_{l+3/2}}^{l+3/2}}^T \\
& \otimes \boxed{\nu_2^{3/2}} \cdots \boxed{\nu_{m_{3/2}}^{3/2}}^T \bigotimes_{j \neq l}^N \underbrace{\boxed{T} \cdots \boxed{T}}_{m_j-l_j} \underbrace{\boxed{L} \cdots \boxed{L}}_{l_j} \bigotimes_{r \neq 3/2, l+3/2}^{N-1/2} \boxed{\nu_1^r} \cdots \boxed{\nu_{m_r}^r}^T \\
& + \sum_{l \geq 1} l l_l \underbrace{\boxed{T} \cdots \boxed{T}}_{m_l-l_l} \underbrace{\boxed{L} \cdots \boxed{L}}_{l_l-1} \otimes \boxed{L} \boxed{\nu_1^{l+3/2}} \cdots \boxed{\nu_{m_{l+3/2}}^{l+3/2}}^T \\
& \otimes \boxed{\nu_2^{3/2}} \cdots \boxed{\nu_{m_{3/2}}^{3/2}}^T \bigotimes_{j \neq l}^N \underbrace{\boxed{T} \cdots \boxed{T}}_{m_j-l_j} \underbrace{\boxed{L} \cdots \boxed{L}}_{l_j} \bigotimes_{r \neq 3/2, l+3/2}^{N-1/2} \boxed{\nu_1^r} \cdots \boxed{\nu_{m_r}^r}^T \\
& + \sum_{i=2}^{m_{3/2}} 3 \underbrace{\boxed{\nu_i^{3/2}} \boxed{T} \cdots \boxed{T}}_{m_3-l_3} \underbrace{\boxed{L} \cdots \boxed{L}}_{l_3} \otimes (-1)^{i+1} \boxed{\nu_2^{3/2}} \cdots \boxed{\hat{\nu}_i^{3/2}} \cdots \boxed{\nu_{m_{3/2}}^{3/2}}^T \\
& \bigotimes_{j \neq 3}^N \underbrace{\boxed{T} \cdots \boxed{T}}_{m_j-l_j} \underbrace{\boxed{L} \cdots \boxed{L}}_{l_j} \bigotimes_{r \neq 3/2}^{N-1/2} \boxed{\nu_1^r} \cdots \boxed{\nu_{m_r}^r}^T \\
& + \sum_{l \geq 2, l \neq 3} \sum_{i=1}^{m_{l-3/2}} \boxed{\nu_i^{l-3/2}} \underbrace{\boxed{T} \cdots \boxed{T}}_{m_l-l_l} \underbrace{\boxed{L} \cdots \boxed{L}}_{l_l} \otimes (-1)^{i+1} \boxed{\nu_1^{l-3/2}} \cdots \boxed{\hat{\nu}_i^{l-3/2}} \cdots \boxed{\nu_{m_{l-3/2}}^{l-3/2}}^T \\
& \otimes \boxed{\nu_2^{3/2}} \cdots \boxed{\nu_{m_{3/2}}^{3/2}}^T \bigotimes_{j \neq l}^N \underbrace{\boxed{T} \cdots \boxed{T}}_{m_j-l_j} \underbrace{\boxed{L} \cdots \boxed{L}}_{l_j} \bigotimes_{r \neq 3/2, l-3/2}^{N-1/2} \boxed{\nu_1^r} \cdots \boxed{\nu_{m_r}^r}^T. \tag{B.23b}
\end{aligned}$$

Next, we will deal with those antisymmetric indices $\{\nu_i^r\}$. In this case, there are much fewer possibilities which we can chosen, i.e.

$$\boxed{\nu_1^r} \cdots \boxed{\nu_{m_r}^r} \equiv \boxed{\nu_1^r} \boxed{\nu_2^r} \left(= \boxed{TL}, \boxed{T}, \boxed{L} \text{ or } \boxed{0} \right). \tag{B.24}$$

Therefore

$$\begin{aligned}
0 = & M \begin{bmatrix} L & \nu_2^{1/2} \end{bmatrix}^T \bigotimes_{j=1}^N \underbrace{\begin{bmatrix} T & \cdots & T & L & \cdots & L \end{bmatrix}}_{m_j-l_j} \bigotimes_{r \neq 1/2}^{N-1/2} \begin{bmatrix} \nu_1^r & \nu_2^r \end{bmatrix}^T \\
& + \sum_{l \geq 1} l(m_l - l_l) \underbrace{\begin{bmatrix} T & \cdots & T & L & \cdots & L \end{bmatrix}}_{m_l-1-l_l} \bigotimes_{l_l} \begin{bmatrix} T & \nu_1^{l+1/2} & \nu_2^{l+1/2} \end{bmatrix}^T \\
& \otimes \begin{bmatrix} \nu_2^{1/2} \end{bmatrix} \bigotimes_{j \neq l}^N \underbrace{\begin{bmatrix} T & \cdots & T & L & \cdots & L \end{bmatrix}}_{m_j-l_j} \bigotimes_{r \neq 1/2, l+1/2}^{N-1/2} \begin{bmatrix} \nu_1^r & \nu_2^r \end{bmatrix}^T \\
& + \sum_{l \geq 1} l l_l \underbrace{\begin{bmatrix} T & \cdots & T & L & \cdots & L \end{bmatrix}}_{m_l-l_l} \bigotimes_{l_l-1} \begin{bmatrix} L & \nu_1^{l+1/2} & \nu_2^{l+1/2} \end{bmatrix}^T \\
& \otimes \begin{bmatrix} \nu_2^{1/2} \end{bmatrix} \bigotimes_{j \neq l}^N \underbrace{\begin{bmatrix} T & \cdots & T & L & \cdots & L \end{bmatrix}}_{m_j-l_j} \bigotimes_{r \neq 1/2, l+1/2}^{N-1/2} \begin{bmatrix} \nu_1^r & \nu_2^r \end{bmatrix}^T \\
& + (-1) \begin{bmatrix} \nu_2^{1/2} & T & \cdots & T & L & \cdots & L \end{bmatrix} \bigotimes_{j \neq 1}^k \underbrace{\begin{bmatrix} T & \cdots & T & L & \cdots & L \end{bmatrix}}_{m_j-l_j} \bigotimes_{r \neq 1/2}^s \begin{bmatrix} \nu_1^r & \nu_2^r \end{bmatrix}^T \\
& + \sum_{l \geq 2} \begin{bmatrix} \nu_1^{l-1/2} & T & \cdots & T & L & \cdots & L \end{bmatrix} \bigotimes_{m_l-l_l} \begin{bmatrix} \nu_2^{l-1/2} \end{bmatrix} \\
& \otimes \begin{bmatrix} \nu_2^{1/2} \end{bmatrix} \bigotimes_{j \neq l}^N \underbrace{\begin{bmatrix} T & \cdots & T & L & \cdots & L \end{bmatrix}}_{m_j-l_j} \bigotimes_{r \neq 1/2, l-1/2}^{N-1/2} \begin{bmatrix} \nu_1^r & \nu_2^r \end{bmatrix}^T \\
& + \sum_{l \geq 2} (-1) \begin{bmatrix} \nu_2^{l-1/2} & T & \cdots & T & L & \cdots & L \end{bmatrix} \bigotimes_{m_l-l_l} \begin{bmatrix} \nu_1^{l-1/2} \end{bmatrix} \\
& \otimes \begin{bmatrix} \nu_2^{1/2} \end{bmatrix} \bigotimes_{j \neq l}^N \underbrace{\begin{bmatrix} T & \cdots & T & L & \cdots & L \end{bmatrix}}_{m_j-l_j} \bigotimes_{r \neq 1/2, l-1/2}^{N-1/2} \begin{bmatrix} \nu_1^r & \nu_2^r \end{bmatrix}^T, \tag{B.25}
\end{aligned}$$

$$\begin{aligned}
0 = & M \begin{bmatrix} L & \nu_2^{3/2} \end{bmatrix}^T \bigotimes_{j=1}^N \underbrace{\begin{bmatrix} T & \cdots & T & L & \cdots & L \end{bmatrix}}_{m_j-l_j} \bigotimes_{r \neq 3/2}^{N-1/2} \begin{bmatrix} \nu_1^r & \nu_2^r \end{bmatrix}^T \\
& + \underbrace{\begin{bmatrix} T & \cdots & T & L & \cdots & L \end{bmatrix}}_{m_1+1-l_1} \bigotimes_{l_1} \begin{bmatrix} T & \nu_1^{1/2} & \nu_2^{1/2} \end{bmatrix}^T \\
& \otimes \begin{bmatrix} \nu_2^{3/2} \end{bmatrix} \bigotimes_{j \neq 1}^N \underbrace{\begin{bmatrix} T & \cdots & T & L & \cdots & L \end{bmatrix}}_{m_j-l_j} \bigotimes_{r \neq 1/2, 3/2}^{N-1/2} \begin{bmatrix} \nu_1^r & \nu_2^r \end{bmatrix}^T \\
& + \sum_{l \geq 1} l (m_l - l_l) \underbrace{\begin{bmatrix} T & \cdots & T & L & \cdots & L \end{bmatrix}}_{m_l-1-l_l} \bigotimes_{l_l} \begin{bmatrix} T & \nu_1^{l+3/2} & \nu_2^{l+3/2} \end{bmatrix}^T \\
& \otimes \begin{bmatrix} \nu_2^{3/2} \end{bmatrix} \bigotimes_{j \neq l}^N \underbrace{\begin{bmatrix} T & \cdots & T & L & \cdots & L \end{bmatrix}}_{m_j-l_j} \bigotimes_{r \neq 3/2, l+3/2}^{N-1/2} \begin{bmatrix} \nu_1^r & \nu_2^r \end{bmatrix}^T \\
& + \sum_{l \geq 1} l l_l \underbrace{\begin{bmatrix} T & \cdots & T & L & \cdots & L \end{bmatrix}}_{m_l-l_l} \bigotimes_{l_l-1} \begin{bmatrix} L & \nu_1^{l+3/2} & \nu_2^{l+3/2} \end{bmatrix}^T \\
& \otimes \begin{bmatrix} \nu_2^{3/2} \end{bmatrix} \bigotimes_{j \neq l}^N \underbrace{\begin{bmatrix} T & \cdots & T & L & \cdots & L \end{bmatrix}}_{m_j-l_j} \bigotimes_{r \neq 3/2, l+3/2}^{N-1/2} \begin{bmatrix} \nu_1^r & \nu_2^r \end{bmatrix}^T \\
& + 3(-1) \underbrace{\begin{bmatrix} \nu_2^{3/2} & T & \cdots & T & L & \cdots & L \end{bmatrix}}_{m_3-l_3} \bigotimes_{j \neq 3}^N \underbrace{\begin{bmatrix} T & \cdots & T & L & \cdots & L \end{bmatrix}}_{m_j-l_j} \bigotimes_{r \neq 3/2}^{N-1/2} \begin{bmatrix} \nu_1^r & \nu_2^r \end{bmatrix}^T \\
& + \sum_{l \geq 2, l \neq 3} \underbrace{\begin{bmatrix} \nu_1^{l-3/2} & T & \cdots & T & L & \cdots & L \end{bmatrix}}_{m_l-l_l} \bigotimes_{l_l} \begin{bmatrix} \nu_2^{l-3/2} \end{bmatrix} \\
& \otimes \begin{bmatrix} \nu_2^{3/2} \end{bmatrix} \bigotimes_{j \neq l}^N \underbrace{\begin{bmatrix} T & \cdots & T & L & \cdots & L \end{bmatrix}}_{m_j-l_j} \bigotimes_{r \neq 3/2, l-3/2}^{N-1/2} \begin{bmatrix} \nu_1^r & \nu_2^r \end{bmatrix}^T \\
& + \sum_{l \geq 2, l \neq 3} \underbrace{\begin{bmatrix} \nu_2^{l-3/2} & T & \cdots & T & L & \cdots & L \end{bmatrix}}_{m_l-l_l} \bigotimes_{l_l} (-1) \begin{bmatrix} \nu_1^{l-3/2} \end{bmatrix} \\
& \otimes \begin{bmatrix} \nu_2^{3/2} \end{bmatrix} \bigotimes_{j \neq l}^N \underbrace{\begin{bmatrix} T & \cdots & T & L & \cdots & L \end{bmatrix}}_{m_j-l_j} \bigotimes_{r \neq 3/2, l-3/2}^{N-1/2} \begin{bmatrix} \nu_1^r & \nu_2^r \end{bmatrix}^T. \tag{B.26}
\end{aligned}$$

Using the following lemma,

Lemma:

$$\begin{bmatrix} T & \cdots & T & L & \cdots & L \end{bmatrix} \bigotimes_{l_1} \underbrace{\begin{bmatrix} T & \cdots & T & L & \cdots & L \end{bmatrix}}_{m_2-l_2} \bigotimes_{m_{1/2}} \underbrace{\begin{bmatrix} \nu_1^{1/2} & \nu_2^{1/2} \end{bmatrix}}_{m_{1/2}} \otimes \begin{bmatrix} \nu_2^{3/2} \end{bmatrix} \otimes \cdots \equiv 0, \tag{B.27}$$

except for (i) $m_{j \geq 3} = m_{r \geq 3/2} = 0$, $l_2 = m_2$, $l_{3/2} = m_{3/2} = 1$ and (ii) $l_1 + l_{1/2} = 2k$ ■

The equations (B.25) and (B.26) reduce to

$$\begin{aligned}
0 = & M \underbrace{\boxed{T \cdots T}}_{m_1-l_1} \underbrace{\boxed{L \cdots L}}_{l_1} \otimes \underbrace{\boxed{L \cdots L}}_{m_2} \otimes \boxed{L \nu_2^{1/2} \nu_3^{1/2}}^T \otimes \boxed{\nu_1^{3/2}} \\
& + l_1 \underbrace{\boxed{T \cdots T}}_{m_1-l_1} \underbrace{\boxed{L \cdots L}}_{l_1-1} \otimes \underbrace{\boxed{L \cdots L}}_{m_2} \otimes \boxed{\nu_2^{1/2} \nu_3^{1/2}}^T \otimes \boxed{L \nu_1^{3/2}}^T \\
& - \boxed{\nu_2^{1/2}} \underbrace{\boxed{T \cdots T}}_{m_1-l_1} \underbrace{\boxed{L \cdots L}}_{l_1} \otimes \underbrace{\boxed{L \cdots L}}_{m_2} \otimes \boxed{\nu_3^{1/2}} \otimes \boxed{\nu_1^{3/2}} \\
& + \boxed{\nu_3^{1/2}} \underbrace{\boxed{T \cdots T}}_{m_1-l_1} \underbrace{\boxed{L \cdots L}}_{l_1} \otimes \underbrace{\boxed{L \cdots L}}_{m_2} \otimes \boxed{\nu_2^{1/2}} \otimes \boxed{\nu_1^{3/2}} \\
& + \underbrace{\boxed{T \cdots T}}_{m_1-l_1} \underbrace{\boxed{L \cdots L}}_{l_1} \otimes \underbrace{\boxed{\nu_1^{3/2} L \cdots L}}_{m_2} \otimes \boxed{\nu_2^{1/2} \nu_3^{1/2}}^T \otimes \boxed{0} \\
& - \underbrace{\boxed{T \cdots T}}_{m_1-l_1} \underbrace{\boxed{L \cdots L}}_{l_1} \otimes \underbrace{\boxed{L \cdots L}}_{m_2} \otimes \boxed{\nu_2^{1/2} \nu_3^{1/2}}^T \otimes \boxed{\nu_1^{3/2}}, \tag{B.28a}
\end{aligned}$$

$$\begin{aligned}
0 = & M \underbrace{\boxed{T \cdots T}}_{m_1-1-l_1} \underbrace{\boxed{L \cdots L}}_{l_1} \otimes \underbrace{\boxed{L \cdots L}}_{m_2} \otimes \boxed{\nu_2^{1/2} \nu_3^{1/2}}^T \otimes \boxed{L \nu_1^{3/2}}^T \\
& + \underbrace{\boxed{T \cdots T}}_{m_1-l_1} \underbrace{\boxed{L \cdots L}}_{l_1} \otimes \underbrace{\boxed{L \cdots L}}_{m_2} \otimes \boxed{T \nu_2^{1/2} \nu_3^{1/2}}^T \otimes \boxed{\nu_1^{3/2}} \\
& - \underbrace{\boxed{T \cdots T}}_{m_1-1-l_1} \underbrace{\boxed{L \cdots L}}_{l_1} \otimes \underbrace{\boxed{\nu_2^{1/2} L \cdots L}}_{m_2} \otimes \boxed{\nu_3^{1/2}} \otimes \boxed{\nu_1^{3/2}} \\
& + \underbrace{\boxed{T \cdots T}}_{m_1-1-l_1} \underbrace{\boxed{L \cdots L}}_{l_1} \otimes \underbrace{\boxed{\nu_3^{1/2} L \cdots L}}_{m_2} \otimes \boxed{\nu_2^{1/2}} \otimes \boxed{\nu_1^{3/2}}. \tag{B.28b}
\end{aligned}$$

From the first equation we have:

For $\nu_2^{1/2} = 0$, $\nu_3^{1/2} = 0$ and $\nu_1^{3/2} = 0$,

$$\begin{aligned}
0 = & M \underbrace{\boxed{T \cdots T}}_{m_1-l_1} \underbrace{\boxed{L \cdots L}}_{l_1} \otimes \underbrace{\boxed{L \cdots L}}_{m_2} \otimes \boxed{L} \otimes \boxed{0} \\
& + l_1 \underbrace{\boxed{T \cdots T}}_{m_1-l_1} \underbrace{\boxed{L \cdots L}}_{l_1-1} \otimes \underbrace{\boxed{L \cdots L}}_{m_2} \otimes \boxed{0} \otimes \boxed{L} \tag{B.29}
\end{aligned}$$

For $\nu_2^{1/2} = T$, $\nu_3^{1/2} = L$ and $\nu_1^{3/2} = 0$,

$$\begin{aligned}
0 &= l_1 \underbrace{T \cdots T}_{m_1-l_1} \underbrace{L \cdots L}_{l_1-1} \otimes \underbrace{L \cdots L}_{m_2} \otimes \begin{bmatrix} T & L \end{bmatrix}^T \otimes \begin{bmatrix} L \end{bmatrix} \\
&\quad - \underbrace{T \cdots T}_{m_1+1-l_1} \underbrace{L \cdots L}_{l_1} \otimes \underbrace{L \cdots L}_{m_2} \otimes \begin{bmatrix} L \end{bmatrix} \otimes \begin{bmatrix} 0 \end{bmatrix} \\
&\quad + \underbrace{T \cdots T}_{m_1-l_1} \underbrace{L \cdots L}_{l_1+1} \otimes \underbrace{L \cdots L}_{m_2} \otimes \begin{bmatrix} T \end{bmatrix} \otimes \begin{bmatrix} 0 \end{bmatrix}
\end{aligned} \tag{B.30}$$

For $\nu_2^{1/2} = T$, $\nu_3^{1/2} = 0$ and $\nu_1^{3/2} = L$,

$$\begin{aligned}
0 &= -M \underbrace{T \cdots T}_{m_1-l_1} \underbrace{L \cdots L}_{l_1} \otimes \underbrace{L \cdots L}_{m_2} \otimes \begin{bmatrix} T & L \end{bmatrix}^T \otimes \begin{bmatrix} L \end{bmatrix} \\
&\quad + (-1) \underbrace{T \cdots T}_{m_1+1-l_1} \underbrace{L \cdots L}_{l_1} \otimes \underbrace{L \cdots L}_{m_2} \otimes \begin{bmatrix} 0 \end{bmatrix} \otimes \begin{bmatrix} L \end{bmatrix} \\
&\quad + \underbrace{T \cdots T}_{m_1-l_1} \underbrace{L \cdots L}_{l_1} \otimes \underbrace{L \cdots L}_{m_2+1} \otimes \begin{bmatrix} T \end{bmatrix} \otimes \begin{bmatrix} 0 \end{bmatrix}
\end{aligned} \tag{B.31}$$

For $\nu_2^{1/2} = L$, $\nu_3^{1/2} = 0$ and $\nu_1^{3/2} = L$,

$$\begin{aligned}
0 &= (-1) \underbrace{T \cdots T}_{m_1-l_1} \underbrace{L \cdots L}_{l_1+1} \otimes \underbrace{L \cdots L}_{m_2} \otimes \begin{bmatrix} 0 \end{bmatrix} \otimes \begin{bmatrix} L \end{bmatrix} \\
&\quad + \underbrace{T \cdots T}_{m_1-l_1} \underbrace{L \cdots L}_{l_1} \otimes \underbrace{L \cdots L}_{m_2+1} \otimes \begin{bmatrix} L \end{bmatrix} \otimes \begin{bmatrix} 0 \end{bmatrix}
\end{aligned} \tag{B.32}$$

From the second equation we have:

For $\nu_2^{1/2} = 0$, $\nu_3^{1/2} = 0$ and $\nu_1^{3/2} = 0$,

$$\begin{aligned}
0 &= M \underbrace{T \cdots T}_{m_1-1-l_1} \underbrace{L \cdots L}_{l_1} \otimes \underbrace{L \cdots L}_{m_2} \otimes \begin{bmatrix} 0 \end{bmatrix} \otimes \begin{bmatrix} L \end{bmatrix} \\
&\quad + \underbrace{T \cdots T}_{m_1-l_1} \underbrace{L \cdots L}_{l_1} \otimes \underbrace{L \cdots L}_{m_2} \otimes \begin{bmatrix} T \end{bmatrix} \otimes \begin{bmatrix} 0 \end{bmatrix}.
\end{aligned} \tag{B.33}$$

For $\nu_2^{1/2} = T$, $\nu_3^{1/2} = L$ and $\nu_1^{3/2} = 0 \Rightarrow$,

$$\begin{aligned}
0 &= M \underbrace{T \cdots T}_{m_1-1-l_1} \underbrace{L \cdots L}_{l_1} \otimes \underbrace{L \cdots L}_{m_2} \otimes \begin{bmatrix} T & L \end{bmatrix}^T \otimes \begin{bmatrix} L \end{bmatrix} \\
&\quad + \underbrace{T \cdots T}_{m_1-1-l_1} \underbrace{L \cdots L}_{l_1} \otimes \underbrace{L \cdots L}_{m_2+1} \otimes \begin{bmatrix} T \end{bmatrix} \otimes \begin{bmatrix} 0 \end{bmatrix}.
\end{aligned} \tag{B.34}$$

For $\nu_2^{1/2} = T$, $\nu_3^{1/2} = 0$ and $\nu_1^{3/2} = L$,

$$0 = 0. \quad (\text{B.35})$$

For $\nu_2^{1/2} = L$, $\nu_3^{1/2} = 0$ and $\nu_1^{3/2} = L$,

$$\begin{aligned} 0 = & \underbrace{\boxed{T} \cdots \boxed{T}}_{m_1-l_1} \underbrace{\boxed{L} \cdots \boxed{L}}_{l_1} \otimes \underbrace{\boxed{L} \cdots \boxed{L}}_{m_2} \otimes \boxed{T} \boxed{L}^T \otimes \boxed{L} \\ & - \underbrace{\boxed{T} \cdots \boxed{T}}_{m_1-1-l_1} \underbrace{\boxed{L} \cdots \boxed{L}}_{l_1} \otimes \underbrace{\boxed{L} \cdots \boxed{L}}_{m_2+1} \otimes \boxed{0} \otimes \boxed{L}. \end{aligned} \quad (\text{B.36})$$

Using the equations (B.29) and (B.32), we get

$$\begin{aligned} & \underbrace{\boxed{T} \cdots \boxed{T}}_{m_1-l_1} \underbrace{\boxed{L} \cdots \boxed{L}}_{l_1-1} \otimes \underbrace{\boxed{L} \cdots \boxed{L}}_{m_2} \otimes \boxed{0} \otimes \boxed{L} \\ & = \frac{l_1!! (-M)^{m_2}}{(l_1 + 2m_2 - 2)!!} \underbrace{\boxed{T} \cdots \boxed{T}}_{m_1-l_1} \underbrace{\boxed{L} \cdots \boxed{L}}_{l_1-1+2m_2} \otimes \boxed{0} \otimes \boxed{0} \otimes \boxed{L}, \end{aligned} \quad (\text{B.37})$$

then using equations (B.31), (B.33) and (B.36), we obtain

$$\begin{aligned} & \underbrace{\boxed{T} \cdots \boxed{T}}_{N-2m_2-2k} \underbrace{\boxed{L} \cdots \boxed{L}}_{2k} \otimes \underbrace{\boxed{L} \cdots \boxed{L}}_{m_2} \otimes \boxed{0} \otimes \boxed{L} \\ & = \left(-\frac{1}{2M}\right)^{m_2} \left(-\frac{1}{2M}\right)^k \frac{(2k-1)!!}{(-M)^k} \underbrace{\boxed{T} \cdots \boxed{T}}_N \otimes \boxed{0} \otimes \boxed{0} \otimes \boxed{L}, \end{aligned} \quad (\text{B.38})$$

the Eq.(B.29) leads to

$$\begin{aligned} & \underbrace{\boxed{T} \cdots \boxed{T}}_{N-2m_2-2k} \underbrace{\boxed{L} \cdots \boxed{L}}_{2k+1} \otimes \underbrace{\boxed{L} \cdots \boxed{L}}_{m_2} \otimes \boxed{L} \otimes \boxed{0} \\ & = \left(-\frac{1}{2M}\right)^{m_2} \left(-\frac{1}{2M}\right)^k \frac{(2k+1)!!}{(-M)^{k+1}} \underbrace{\boxed{T} \cdots \boxed{T}}_N \otimes \boxed{0} \otimes \boxed{0} \otimes \boxed{L}, \end{aligned} \quad (\text{B.39})$$

the Eq.(B.33) leads to

$$\begin{aligned} & \underbrace{\boxed{T} \cdots \boxed{T}}_{N-2m_2-2k+1} \underbrace{\boxed{L} \cdots \boxed{L}}_{2k} \otimes \underbrace{\boxed{L} \cdots \boxed{L}}_{m_2} \otimes \boxed{T} \otimes \boxed{0} \\ & = \left(-\frac{1}{2M}\right)^{m_2} \left(-\frac{1}{2M}\right)^k \frac{(2k-1)!!}{(-M)^{k-1}} \underbrace{\boxed{T} \cdots \boxed{T}}_N \otimes \boxed{0} \otimes \boxed{0} \otimes \boxed{L}, \end{aligned} \quad (\text{B.40})$$

the Eq.(B.36) leads to

$$\begin{aligned}
& \underbrace{\boxed{T \cdots T}}_{N-2m_2-2k+1} \underbrace{\boxed{L \cdots L}}_{2k} \otimes \underbrace{\boxed{L \cdots L}}_{m_2-1} \otimes \boxed{T L}^T \otimes \boxed{L} \\
&= \left(-\frac{1}{2M}\right)^{m_2} \left(-\frac{1}{2M}\right)^k \frac{(2k-1)!!}{(-M)^k} \underbrace{\boxed{T \cdots T}}_N \otimes \boxed{0} \otimes \boxed{0} \otimes \boxed{L}.
\end{aligned} \tag{B.41}$$

We solve the above equations, the ratios between the physical states in the NS sector in the high energy limit are given as

$$\begin{aligned}
& \underbrace{\boxed{T \cdots T L \cdots L}}_{N-2m_2-2k} \underbrace{\boxed{L \cdots L}}_{2k} \otimes \underbrace{\boxed{L \cdots L}}_{m_2} \otimes \boxed{0} \otimes \boxed{L} \\
&= \left(-\frac{1}{2M}\right)^{m_2} \left(-\frac{1}{2M}\right)^k \frac{(2k-1)!!}{(-M)^k} \underbrace{\boxed{T \cdots T}}_N \otimes \boxed{0} \otimes \boxed{0} \otimes \boxed{L},
\end{aligned} \tag{B.42}$$

$$\begin{aligned}
& \underbrace{\boxed{T \cdots T L \cdots L}}_{N-2m_2-2k} \underbrace{\boxed{L \cdots L}}_{2k+1} \otimes \underbrace{\boxed{L \cdots L}}_{m_2} \otimes \boxed{L} \otimes \boxed{0} \\
&= \left(-\frac{1}{2M}\right)^{m_2} \left(-\frac{1}{2M}\right)^k \frac{(2k+1)!!}{(-M)^{k+1}} \underbrace{\boxed{T \cdots T}}_N \otimes \boxed{0} \otimes \boxed{0} \otimes \boxed{L},
\end{aligned} \tag{B.43}$$

$$\begin{aligned}
& \underbrace{\boxed{T \cdots T}}_{N-2m_2-2k+1} \underbrace{\boxed{L \cdots L}}_{2k} \otimes \underbrace{\boxed{L \cdots L}}_{m_2} \otimes \boxed{T} \otimes \boxed{0} \\
&= \left(-\frac{1}{2M}\right)^{m_2} \left(-\frac{1}{2M}\right)^k \frac{(2k-1)!!}{(-M)^{k-1}} \underbrace{\boxed{T \cdots T}}_N \otimes \boxed{0} \otimes \boxed{0} \otimes \boxed{L},
\end{aligned} \tag{B.44}$$

$$\begin{aligned}
& \underbrace{\boxed{T \cdots T}}_{N-2m_2-2k+1} \underbrace{\boxed{L \cdots L}}_{2k} \otimes \underbrace{\boxed{L \cdots L}}_{m_2-1} \otimes \boxed{T L}^T \otimes \boxed{L} \\
&= \left(-\frac{1}{2M}\right)^{m_2} \left(-\frac{1}{2M}\right)^k \frac{(2k-1)!!}{(-M)^k} \underbrace{\boxed{T \cdots T}}_N \otimes \boxed{0} \otimes \boxed{0} \otimes \boxed{L},
\end{aligned} \tag{B.45}$$

Appendix C: Kinematic relations in the RR

In this appendix, we list the expressions of the kinematic variables we used in the evaluation of 4-point functions in this paper. For convenience, we take the center of momentum

frame and choose the momenta of particles 1 and 2 to be along the X^1 -direction. The high energy scattering plane is defined to be on the $X^1 - X^2$ plane.

The momenta of the four particles are

$$k_1 = \left(+\sqrt{p^2 + M_1^2}, -p, 0 \right), \quad (\text{C.1})$$

$$k_2 = \left(+\sqrt{p^2 + M_2^2}, +p, 0 \right), \quad (\text{C.2})$$

$$k_3 = \left(-\sqrt{q^2 + M_3^2}, -q \cos \theta, -q \sin \theta \right), \quad (\text{C.3})$$

$$k_4 = \left(-\sqrt{q^2 + M_4^2}, +q \cos \theta, +q \sin \theta \right) \quad (\text{C.4})$$

where $p \equiv |\tilde{p}|$, $q \equiv |\tilde{q}|$ and $k_i^2 = -M_i^2$. In the calculation of the string scattering amplitudes, we use the following formulas

$$-k_1 \cdot k_2 = \sqrt{p^2 + M_1^2} \cdot \sqrt{p^2 + M_2^2} + p^2 = \frac{1}{2} (s - M_1^2 - M_2^2), \quad (\text{C.5})$$

$$-k_2 \cdot k_3 = -\sqrt{p^2 + M_2^2} \cdot \sqrt{q^2 + M_3^2} + pq \cos \theta = \frac{1}{2} (t - M_2^2 - M_3^2), \quad (\text{C.6})$$

$$-k_1 \cdot k_3 = -\sqrt{p^2 + M_1^2} \cdot \sqrt{q^2 + M_3^2} - pq \cos \theta = \frac{1}{2} (u - M_1^2 - M_3^2) \quad (\text{C.7})$$

where the Mandelstam variables are defined as usual with

$$s + t + u = \sum_i M_i^2 = 2N - 1. \quad (\text{C.8})$$

The center of mass energy E is defined as

$$E = \frac{1}{2} \left(\sqrt{p^2 + M_1^2} + \sqrt{p^2 + M_2^2} \right) = \frac{1}{2} \left(\sqrt{q^2 + M_3^2} + \sqrt{q^2 + M_4^2} \right). \quad (\text{C.9})$$

We define the polarizations of the string state on the scattering plane as

$$e^P = \frac{1}{M_2} \left(\sqrt{p^2 + M_2^2}, p, 0 \right), \quad (\text{C.10})$$

$$e^L = \frac{1}{M_2} \left(p, \sqrt{p^2 + M_2^2}, 0 \right), \quad (\text{C.11})$$

$$e^T = (0, 0, 1). \quad (\text{C.12})$$

The projections of the momenta on the scattering plane can be calculated as (here we only list the ones we need for our calculations)

$$e^P \cdot k_1 = -\frac{1}{M_2} \left(\sqrt{p^2 + M_1^2} \sqrt{p^2 + M_2^2} + p^2 \right), \quad (\text{C.13})$$

$$e^L \cdot k_1 = -\frac{p}{M_2} \left(\sqrt{p^2 + M_1^2} + \sqrt{p^2 + M_2^2} \right), \quad (\text{C.14})$$

$$e^T \cdot k_1 = 0 \quad (\text{C.15})$$

and

$$e^P \cdot k_3 = \frac{1}{M_2} \left(\sqrt{q^2 + M_3^2} \sqrt{p^2 + M_2^2} - pq \cos \theta \right), \quad (\text{C.16})$$

$$e^L \cdot k_3 = \frac{1}{M_2} \left(p \sqrt{q^2 + M_3^2} - q \sqrt{p^2 + M_2^2} \cos \theta \right), \quad (\text{C.17})$$

$$e^T \cdot k_3 = -q \sin \theta. \quad (\text{C.18})$$

We now expand the kinematic relations to the subleading orders in the RR. We first express all kinematic variables in terms of s and t , and then expand all relevant quantities in s :

$$E_1 = \frac{s - (M_2^2 + 2)}{2\sqrt{2}}, \quad (\text{C.19})$$

$$E_2 = \frac{s + (M_2^2 + 2)}{2\sqrt{2}}, \quad (\text{C.20})$$

$$|\mathbf{k}_2| = \sqrt{E_1^2 + 2}, \quad |\mathbf{K}_3| = \sqrt{\frac{s}{4} + 2}; \quad (\text{C.21})$$

$$e_P \cdot k_1 = -\frac{1}{2M_2} s + \left(-\frac{1}{M_2} + \frac{M_2}{2} \right), \quad (\text{exact}) \quad (\text{C.22})$$

$$\begin{aligned} e_L \cdot k_1 = & -\frac{1}{2M_2} s + \left(-\frac{1}{M_2} + \frac{M_2}{2} \right) - 2M_2 s^{-1} - 2M_2(M_2^2 - 2)s^{-2} \\ & - 2m_2(M_2^4 - 6M_2^2 + 4)s^{-3} - 2M_2(M_2^6 - 12M_2^4 + 24M_2^2 - 8)s^{-4} + O(s^{-5}), \end{aligned} \quad (\text{C.23})$$

$$e_T \cdot k_1 = 0. \quad (\text{C.24})$$

A key step is to express the scattering angle θ in terms of s and t . This can be achieved by solving

$$t = - \left(-\left(E_2 - \frac{\sqrt{s}}{2}\right)^2 + (|\mathbf{k}_2| - |\mathbf{K}_3| \cos \theta)^2 + |\mathbf{K}_3|^2 \sin^2 \theta \right) \quad (\text{C.25})$$

to obtain

$$\theta = \arccos \left(\frac{s + 2t - M_2^2 + 6}{\sqrt{s+8} \sqrt{\frac{(s+2)^2 - 2(s-2)M_2^2 + M_2^4}{s}}} \right). \quad (\text{exact}) \quad (\text{C.26})$$

One can then calculate the following expansions which we used in the subleading order calculation in section V

$$e_P \cdot k_3 = \frac{1}{M_2} (E_2 \frac{\sqrt{s}}{2} - |\mathbf{k}_2| |\mathbf{k}_3| \cos \theta) = -\frac{t + 2 - M_2^2}{2M_2}, \quad (\text{C.27})$$

$$\begin{aligned} e_L \cdot k_3 &= \frac{1}{M_2} (k_2 \frac{\sqrt{2}}{2} - E_2 k_3 \cos \theta) \\ &= -\frac{t + 2 + M_2^2}{2M_2} - M_2 t s^{-1} - M_2 [-4(t+1) + M_2^2(t-2)] s^{-2} \\ &\quad - M_2 [4(4+3t) - 12tM_2^2 + (t-4)M_2^4] s^{-3} - M_2 [-16(3+2t) + 24(2+3t)M_2^2 \\ &\quad - 24(-1+t)M_2^4 + (-6+t)M_2^6] s^{-4} + O(s^{-5}), \end{aligned} \quad (\text{C.28})$$

$$\begin{aligned} e_T \cdot k_3 &= -|\mathbf{k}_3| \sin \theta \\ &= -\sqrt{-t} - \frac{1}{2} \sqrt{-t} (2 + t + M_2^2) s^{-1} \\ &\quad - \frac{1}{8\sqrt{-t}} [32 + 52t + 20t^2 + t^3 + (32 + 20t - 6t^2)M_2^2 + (8 - 3t)M_2^4] s^{-2} \\ &\quad + \frac{1}{16\sqrt{-t}} [320 + 456t + 188t^2 + 22t^3 + t^4 - (-224 + 36t + 132t^2 + 5t^3)M_2^2 \\ &\quad + (-16 - 122t + 15t^2)M_2^4 + (-24 + 5t)M_2^6] s^{-3} \\ &\quad + \frac{1}{128(-t)^{3/2}} [1024 + 12032t + 16080t^2 + 7520t^3 + 1432t^4 + 136t^5 + 5t^6 \\ &\quad - 4(-512 - 896t + 2232t^2 + 1844t^3 + 170t^4 + 7t^5)M_2^2 \\ &\quad + 2(768 - 2240t - 2372t^2 + 1172t^3 + 35t^4)M_2^4 \\ &\quad - 4(-128 + 288t - 450t^2 + 35t^3)M_2^6 + (64 + 240t - 35t^2)M_2^8] s^{-4} + O(s^{-5}). \end{aligned} \quad (\text{C.29})$$

Appendix D: Recurrence relations of Kummer functions

In this appendix, we review the recurrence relations of Kummer functions of the second kind [71]. The Kummer function of the second kind U is defined to be

$$U(a, c, x) = \frac{\pi}{\sin \pi c} \left[\frac{M(a, c, x)}{(a-c)!(c-1)!} - \frac{x^{1-c} M(a+1-c, 2-c, x)}{(a-1)!(1-c)!} \right] \quad (c \neq 2, 3, 4, \dots) \quad (\text{D.1})$$

where $M(a, c, x) = \sum_{j=0}^{\infty} \frac{(a)_j}{(c)_j} \frac{x^j}{j!}$ is the Kummer function of the first kind. Here $(a)_j = a(a+1)(a+2)\dots(a+j-1)$ is the Pochhammer symbol. U and M are the two solutions of the Kummer Equation

$$xy''(x) + (c-x)y'(x) - ay(x) = 0. \quad (\text{D.2})$$

For any confluent hypergeometric function with parameters (a, c) the four functions with parameters $(a-1, c)$, $(a+1, c)$, $(a, c-1)$ and $(a, c+1)$ are called the contiguous functions.

It follows, from the Kummer Equation Eq.(D.2) and derivatives of Kummer functions

$$U(a+1, c+1, x) = \frac{-1}{a}U'(a, c, x), \quad (\text{D.3})$$

$$U(a+1, c, x) = \frac{1}{1+a-c}U(a, c, x) + \frac{x}{a(1+a-c)}U'(a, c, x), \quad (\text{D.4})$$

$$U(a, c-1, x) = \frac{1-c}{1+a-c}U(a, c, x) - \frac{x}{1+a-c}U'(a, c, x), \quad (\text{D.5})$$

$$U(a, c+1, x) = U'(a, c, x) - U'(a, c, x), \quad (\text{D.6})$$

$$U(a-1, c, x) = (x+a-c)U(a, c, x) - xU'(a, c, x), \quad (\text{D.7})$$

$$U(a-1, c-1, x) = (1+x-c)U(a, c, x) - xU'(a, c, x), \quad (\text{D.8})$$

that a recurrence relation exists between any such function and any two of its contiguous functions. There are six recurrence relations

$$U(a-1, c, x) - (2a-c+x)U(a, c, x) + a(1+a-c)U(a+1, c, x) = 0, \quad (\text{D.9})$$

$$(c-a-1)U(a, c-1, x) - (x+c-1)U(a, c, x) + xU(a, c+1, x) = 0, \quad (\text{D.10})$$

$$U(a, c, x) - aU(a+1, c, x) - U(a, c-1, x) = 0, \quad (\text{D.11})$$

$$(c-a)U(a, c, x) + U(a-1, c, x) - xU(a, c+1, x) = 0, \quad (\text{D.12})$$

$$(a+x)U(a, c, x) - xU(a, c+1, x) + a(c-a-1)U(a+1, c, x) = 0, \quad (\text{D.13})$$

$$(a+x-1)U(a, c, x) - U(a-1, c, x) + (1+a-c)U(a, c-1, x) = 0. \quad (\text{D.14})$$

From any two of these six relations the remaining four recurrence relations can be deduced. Thus they are not independent. For example, one can deduces recurrence relation Eq.(D.9) from Eq.(D.11) and Eq.(D.12). We start with Eq.(D.11) with $c \rightarrow c+1$

$$U(a, c+1, x) - aU(a+1, c+1, x) - U(a, c, x) = 0. \quad (\text{D.15})$$

We consider Eq.(D.12)+ x ·Eq.(D.15) to deduce

$$(c-a-x)U(a, c, x) + U(a-1, c, x) - axU(a+1, c+1, x) = 0. \quad (\text{D.16})$$

Next we replace Eq.(D.12) with $a \rightarrow a + 1$ to get

$$(c - a - 1) U(a + 1, c, x) + U(a, c, x) - xU(a + 1, c + 1, x) = 0 . \quad (\text{D.17})$$

Finally we consider Eq.(D.16)–a·Eq.(D.17) to deduce

$$(c - 2a - x) U(a, c, x) + U(a - 1, c, x) - a(c - a - 1) U(a + 1, c, x) = 0 , \quad (\text{D.18})$$

which is nothing but Eq.(D.9).

The confluent hypergeometric function with parameters $(a \pm m, c \pm n)$ for $m, n = 0, 1, 2, \dots$ are called associated functions. Again it can be shown that there exist relations between any three associated functions, so that any confluent hypergeometric function can be expressed in terms of any two of its associated functions.

Appendix E: Regge string ZNS

There are two types of ZNS in the old covariant first quantized string spectrum

$$\text{Type I : } L_{-1} |x\rangle , \text{ where } L_1 |x\rangle = L_2 |x\rangle = 0, \quad L_0 |x\rangle = 0; \quad (\text{E.1})$$

$$\text{Type II : } (L_{-2} + \frac{3}{2}L_{-1}^2) |\tilde{x}\rangle , \text{ where } L_1 |\tilde{x}\rangle = L_2 |\tilde{x}\rangle = 0, \quad (L_0 + 1) |\tilde{x}\rangle = 0. \quad (\text{E.2})$$

Eq.(E.1) and Eq.(E.2) can be derived from Kac determinant in conformal field theory. While type I states have zero-norm at any spacetime dimension, type II states have zero-norm *only* at D=26. The existence of type II ZNS signals the importance of ZNS in the structure of the theory of string. In fact, the linear relations obtained by high energy limit of stringy Ward identities or decoupling of ZNS in the GR were just good enough to solve all the high energy amplitudes in terms of one amplitude.

In the RR, however, the Regge stringy Ward identities or decoupling of ZNS in the RR turned out to be not good enough to solve all the Regge scattering amplitudes algebraically. This is due to the much more numerous Regge string scattering amplitudes than those in the GR at each fixed mass level. In this appendix, we list all ZNS for $M^2 = 2$ and 4 and calculate their Regge limit which we use in the text to demonstrate the calculation. At the first massive level $k^2 = -2$, there is a type II ZNS

$$[\frac{1}{2}\alpha_{-1} \cdot \alpha_{-1} + \frac{5}{2}k \cdot \alpha_{-2} + \frac{3}{2}(k \cdot \alpha_{-1})^2] |0, k\rangle \quad (\text{E.3})$$

and a type I ZNS

$$[\theta \cdot \alpha_{-2} + (k \cdot \alpha_{-1})(\theta \cdot \alpha_{-1})] |0, k\rangle, \theta \cdot k = 0. \quad (\text{E.4})$$

In the Regge limit, the polarizations of the 2nd particle with momentum k_2 on the scattering plane used in the text were defined to be $e^P = \frac{1}{M_2}(E_2, k_2, 0) = \frac{k_2}{M_2}$ as the momentum polarization, $e^L = \frac{1}{M_2}(k_2, E_2, 0)$ the longitudinal polarization and $e^T = (0, 0, 1)$ the transverse polarization which lies on the scattering plane. $\eta_{\mu\nu} = \text{diag}(-1, 1, 1)$. The three vectors e^P , e^L and e^T satisfy the completeness relation $\eta_{\mu\nu} = \sum_{\alpha, \beta} e_\mu^\alpha e_\nu^\beta \eta_{\alpha\beta}$ where $\mu, \nu = 0, 1, 2$ and $\alpha, \beta = P, L, T$ and $\alpha_{-1}^T = \sum_\mu e_\mu^T \alpha_{-1}^\mu$, $\alpha_{-1}^T \alpha_{-2}^L = \sum_{\mu, \nu} e_\mu^T e_\nu^L \alpha_{-1}^\mu \alpha_{-2}^\nu$ etc.

In the Regge limit, the type II ZNS in Eq.(E.3) gives the Regge string zero-norm state (RZNS)

$$(\sqrt{2}\alpha_{-2}^P - \alpha_{-1}^P \alpha_{-1}^P - \frac{1}{5}\alpha_{-1}^L \alpha_{-1}^L - \frac{1}{5}\alpha_{-1}^T \alpha_{-1}^T) |0, k\rangle. \quad (\text{E.5})$$

Type I ZNS in Eq.(E.4) gives two RZNS

$$(\alpha_{-2}^T - \sqrt{2}\alpha_{-1}^P \alpha_{-1}^T) |0, k\rangle, \quad (\text{E.6})$$

$$(\alpha_{-2}^L - \sqrt{2}\alpha_{-1}^P \alpha_{-1}^L) |0, k\rangle \quad (\text{E.7})$$

RZNS in Eq.(E.6) and Eq.(E.7) correspond to choose $\theta^\mu = e^T$ and $\theta^\mu = e^L$ respectively. Note that the norms of Regge "zero-norm" states may not be zero. For instance the norm of Eq.(E.5) is not zero. They are just used to produce Regge stringy Ward identities Eq.(11.91), Eq.(11.89) and Eq.(11.90) in the text.

At the second massive level $k^2 = -4$, there is a type I scalar ZNS

$$\begin{aligned} & [\frac{17}{4}(k \cdot \alpha_{-1})^3 + \frac{9}{2}(k \cdot \alpha_{-1})(\alpha_{-1} \cdot \alpha_{-1}) + 9(\alpha_{-1} \cdot \alpha_{-2}) \\ & + 21(k \cdot \alpha_{-1})(k \cdot \alpha_{-2}) + 25(k \cdot \alpha_{-3})] |0, k\rangle, \end{aligned} \quad (\text{E.8})$$

a symmetric type I spin two ZNS

$$[2\theta_{\mu\nu}\alpha_{-1}^{(\mu}\alpha_{-2}^{\nu)} + k_\lambda\theta_{\mu\nu}\alpha_{-1}^{\lambda\mu\nu}] |0, k\rangle, k \cdot \theta = \eta^{\mu\nu}\theta_{\mu\nu} = 0, \theta_{\mu\nu} = \theta_{\nu\mu} \quad (\text{E.9})$$

where $\alpha_{-1}^{\lambda\mu\nu} \equiv \alpha_{-1}^\lambda \alpha_{-1}^\mu \alpha_{-1}^\nu$ and two vector ZNS

$$[(\frac{5}{2}k_\mu k_\nu \theta'_\lambda + \eta_{\mu\nu} \theta'_\lambda) \alpha_{-1}^{(\mu\nu\lambda)} + 9k_\mu \theta'_\nu \alpha_{-1}^{(\mu\nu)} + 6\theta'_\mu \alpha_{-1}^\mu] |0, k\rangle, \theta \cdot k = 0, \quad (\text{E.10})$$

$$[(\frac{1}{2}k_\mu k_\nu \theta_\lambda + 2\eta_{\mu\nu} \theta_\lambda) \alpha_{-1}^{(\mu\nu\lambda)} + 9k_\mu \theta_\nu \alpha_{-1}^{[\mu\nu]} - 6\theta_\mu \alpha_{-1}^\mu] |0, k\rangle, \theta \cdot k = 0. \quad (\text{E.11})$$

Note that Eq.(E.10) and Eq.(E.11) are linear combinations of a type I and a type II ZNS. This completes the four ZNS at the second massive level $M^2 = 4$.

In the Regge limit, the scalar ZNS in Eq.(E.8) gives the RZNS

$$[25(\alpha_{-1}^P)^3 + 9\alpha_{-1}^P(\alpha_{-1}^L)^2 + 9\alpha_{-1}^P(\alpha_{-1}^T)^2 - 9\alpha_{-2}^L \alpha_{-1}^L - 9\alpha_{-2}^T \alpha_{-1}^T - 75\alpha_{-2}^P \alpha_{-1}^P + 50\alpha_{-3}^P] |0, k\rangle. \quad (\text{E.12})$$

For the type I spin two ZNS in Eq.(E.9), we define $\theta_{\mu\nu} = \sum_{\alpha,\beta} e_\mu^\alpha e_\nu^\beta u_{\alpha\beta}$, symmetric and transverse conditions on $\theta_{\mu\nu}$ implies

$$u_{\alpha\beta} = u_{\beta\alpha}; u_{PP} = u_{PL} = u_{PT} = 0. \quad (\text{E.13})$$

Naively, the traceless condition on $\theta_{\mu\nu}$ implies

$$u_{PP} - u_{LL} - u_{TT} = 0. \quad (\text{E.14})$$

However, for the reason which will become clear later that one needs to include at least one component u_{NN} perpendicular to the scattering plane and modify Eq.(E.14) to

$$u_{PP} - u_{LL} - u_{TT} - u_{NN} = 0. \quad (\text{E.15})$$

Note that, in the Regge limit, Eq.(E.15) reduces to Eq.(E.14). However, the solutions for Eq.(E.13) and Eq.(E.15) give three RZNS

$$(\alpha_{-1}^L \alpha_{-2}^L - \alpha_{-1}^P \alpha_{-1}^L \alpha_{-1}^L) |0, k\rangle, \quad (\text{E.16})$$

$$(\alpha_{-1}^T \alpha_{-2}^T - \alpha_{-1}^P \alpha_{-1}^T \alpha_{-1}^T) |0, k\rangle, \quad (\text{E.17})$$

$$(\alpha_{-1}^{(L} \alpha_{-2}^{T)} - \alpha_{-1}^P \alpha_{-1}^L \alpha_{-1}^T) |0, k\rangle, \quad (\text{E.18})$$

while Eq.(E.13) and Eq.(E.14) give only two RZNS

$$(\alpha_{-1}^L \alpha_{-2}^L - \alpha_{-1}^P \alpha_{-1}^L \alpha_{-1}^L - \alpha_{-1}^T \alpha_{-2}^T + \alpha_{-1}^P \alpha_{-1}^T \alpha_{-1}^T) |0, k\rangle \quad (\text{E.19})$$

and Eq.(E.18). Note that Eq.(E.19) is just a linear combination of Eq.(E.16) and Eq.(E.17). For the high energy fixed angle calculation in [27, 28], the corresponding extra ZNS will not affect the final result there. The vector ZNS in Eq.(E.10) gives two RZNS

$$[6\alpha_{-3}^T - 18\alpha_{-1}^{(P} \alpha_{-2}^{T)} + 9\alpha_{-1}^P \alpha_{-1}^P \alpha_{-1}^T + \alpha_{-1}^L \alpha_{-1}^L \alpha_{-1}^T + \alpha_{-1}^T \alpha_{-1}^T \alpha_{-1}^T] |0, k\rangle, \quad (\text{E.20})$$

$$[6\alpha_{-3}^L - 18\alpha_{-1}^{(P)}\alpha_{-2}^{(L)} + 9\alpha_{-1}^P\alpha_{-1}^P\alpha_{-1}^L + \alpha_{-1}^L\alpha_{-1}^L\alpha_{-1}^L + \alpha_{-1}^L\alpha_{-1}^T\alpha_{-1}^T]|0, k\rangle. \quad (\text{E.21})$$

The vector ZNS in Eq.(E.11) gives two RZNS

$$[3\alpha_{-3}^T + 9\alpha_{-1}^{[P}\alpha_{-2}^{T]} - \alpha_{-1}^L\alpha_{-1}^L\alpha_{-1}^T - \alpha_{-1}^T\alpha_{-1}^T\alpha_{-1}^T]|0, k\rangle, \quad (\text{E.22})$$

$$[3\alpha_{-3}^L + 9\alpha_{-1}^{[P}\alpha_{-2}^{L]} - \alpha_{-1}^L\alpha_{-1}^L\alpha_{-1}^L - \alpha_{-1}^L\alpha_{-1}^T\alpha_{-1}^T]|0, k\rangle. \quad (\text{E.23})$$

There are totally 8 RZNS at the mass level $M^2 = 4$.

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